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6. AUTHOR(S)
Benqi Guo

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Dept. of Mathematics, Univ. of Manitoba
Winnipeg, Manitoba R3T 2N2, Canada

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13. ABSTRACT (Maximum 200 words)

This final progress report summarizes the progresses of our research in the period 1999-2002. Starting with problems of periodic lattices in entire spaces, and we focus on the mathematical framework for problems of unstructured lattice in unbounded domains without absolute terms. In this frmaework, the existence and uniqueness of solution for the problem without absolute terms in entire and half spaces $\mathbf{R}^d$ and $\mathbf{R}^d_+, d = 1, 2, 3$ are proved in energy spaces. For unstructured lattices, new methodlogy and approach have been developed successfully, i.e. extension of grid functions by linear interpolaton, which is essntial to the some embedding results in discrete Sobolev spaces. These embeddings lead to the proof of existence of solutions.

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Mathematical Framework For Problems of Unbounded Lattices

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Benqi Guo

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Department of Mathematics

University of Manitoba

Winnipeg, MB R3T 2N2, Canada

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1. INTRODUCTION

Lattice materials are porous materials consisting of periodic cells or non-periodic cells. The cells are composed of rods, or shells, or solid structures. The size of cell is usually small with respect to the size of the body filled with the lattice materials. The lattice materials with simple microstructures are characterized by a single length scale, for instance, Lattice Block Materials which are developed by JAMCORP corporation. The hierarchic lattice materials have hierarchic multi-scale microstructures. In either case, we deal with a multi-scale problem. The lattice materials can offer significantly higher strength-to-weight and stiffness-to-weight ratio than their base materials and have obvious advantages in engineering applications. Besides heterogeneous materials[17, 28, 30, 33], lattice models are used in many other applications such as porous media[12, 16, 33], fracture models[31], crystal physics[21], biophysics[20]. For a survey of some applications, we refer to [30, 32].

Various mechanical models for the lattice problems with periodic and non-periodic structures have been studied. There are papers addressing these problems, especially in the mechanics, material science, and physics literatures. For mathematical theory which is related to the problem of the lattice materials we refer to the book [8] and her various papers, e.g. [7, 9, 10]. Various mathematical aspects of lattice problems have been studied. Problems of periodic lattices in entire spaces and numerical methods are analyzed based on Greens function and Fourier transform [3, 21, 22, 23, 24]. A combination of homogenization and multigrid method on unstructured mesh is applied to problems of non-periodic lattices [6, 14]. The recovery method has been developed recently for problems of bounded lattice [4], which convert discrete lattice equations to a finite element discretization of a continuous partial differential equation.

The results of these papers are mostly for problems of periodic lattices in entire spaces $\mathbf{R}^d$, but not for problems of unstructured lattices on unbounded and bounded domains with prescribed boundary conditions. In addition, the scale of cells in these papers is assumed so small that asymptotic arguments such as homogenization can be utilized. Further, most papers only address the problems in presence of absolute terms in the equations so that the corresponding bilinear form satisfies the inf-sup condition on a pair of discrete Sobolev spaces. In practical applications, these assumptions may not be valid. If the scale of cells may not extremely small, then the homogenization will not give us satisfactory solution of lattice problems. In particular, the absolute terms do not exist in applications, it is a pure mathematical hypothesis for the convenience in mathematical analysis. For problems without such absolute terms we need to develop new mathematical approaches for problem of unstructured lattices.

This final progress report summarizes the progresses of our research in the period 1999-2002, including many results which are original and have not been published yet. In the past three years, we started with problems of periodic lattices in entire spaces, and then focused on the mathematical framework for problems of unstructured lattice in unbounded domains without absolute

terms. The Fourier transform is a very powerful tool for study of problems of lattices with periodic structure and for design of effective numerical methods such as generalized p-version of finite element methods [25, 26], even for those on bounded and unbounded domains with boundary layers. But it is not valid for unstructured lattices. The new approach is to establish the equivalence between the problems of lattices and partial differential equations(not homogenized equations), and the equivalence between grid functions and continuous functions. With help of such equivalences we can establish various embedding results among discrete function spaces which lead to the existence and uniqueness of solution of lattice problems, also it will results in an effective algorithm via the numerical solution of the corresponding partial differential equations, e.g. multigrid method. The problems we presented are of truss type which result in systems of difference equations with infinite number of unknowns. Such lattice problems are addressed in discrete energy space or Sobolev spaces over lattices. The mathematical framework we have established and mathematical foundation we have laid down for the problems of unstructured trusses can be applied to or generalized to problems with complicated microstructures such as plates and shells, or three dimensional solid structures and many non-periodic lattice problems.

The report is organized as the follows. In Section 2, we address problems of periodic trusses in entire spaces $\mathbf{R}^d, d = 1, 2, 3$. The problem is setup with various important concepts such as connectivity and rigidity. With Fourier transformation the equivalence between a discrete problem and a semi-discrete problem is established, and a representation formula of solution of truss problem is derived. at the end of this section, two lattice problems are analyzed in the theoretical framework. In next section we present the major progresses of research on the problems of unstructured trusses without absolute terms in entire spaces. The extension of grid functions to continuous functions over $\mathbf{R}^d$ is defined by a linear interpolation is explicitly constructed, which leads to an embedding of the energy space into weighted L^2 spaces. The existence and uniqueness of the solution are proved for problem without absolute terms in one, two and three dimensions by different way. In Section 4 a boundary value problem of unstructured trusses in half spaces $\mathbf{R}^d_+$ is analyzed. A sufficient condition on the external force f is derived, under which the existence and uniqueness of solution of the boundary value problem is proved. At last we conclude with current and future research directions on problems of lattices, mathematical models and computational methods. In two appendices, the solution of Possion equation in entire spaces $\mathbf{R}^d, d = 2, 3$ and the boundary value problem in half spaces $\mathbf{R}^d, d = 2, 3$ are discussed in modified function spaces, which are parallel to the problems of lattices in entire and half spaces, some embedding results are essential to the existence of the solution of such lattice problems.

2. PERIODIC LATTICES IN ENTIRE SPACES

2.1. General setting of periodic in entire spaces $\mathbf{R}^d$

Lattices are comprised of cells and nodes. Let Q be a master cell in $\mathbf{R}^d$ with unit size, which is an interval in one dimension, and is a polygon in two dimensions, and is an polyhedron in three dimensions. The master cell is extended periodically in entire space by an integer translation:

$$Q_m = \{y \in \mathbf{R}^d | y = x + \sum_{i=1}^{d} m_i t^{(i)}, x \in Q\}, m \in \mathcal{Z}^d. \tag{2.1}$$

where $\mathcal{Z} = \{0, \pm 1, \pm 2, \dots\}$ denotes the set of all integers, and $t^{(i)}$ is a unit vector in x_i axis. There is a set K_Q of nodes $\{x^{(\kappa)}\}_{\kappa=1}^{q}$ in the master cell Q, and a set K_m is the integer translation of K_Q by

$$K_m = \{x^{(m,\kappa)} = x^{(\kappa)} + \sum_{i=1}^{d} m_i t^{(i)}, x^{(\kappa)} \in K_Q\}, m \in \mathcal{Z}^d. \tag{2.2}$$

Note that the indices κ of nodes in each cell Q_m are the same although the locations of these points in different cells are different. Hence we denote the set of indices $\{1, 2, \dots, q\}$ by $\mathcal{K}$. Without losing generality, we assume that the cells Q_m's and sets K_m's are mutually disjoint, namely,

$$Q_n \cap Q_m = \emptyset, K_n \cap K_m = \emptyset, n \neq m, n, m \in \mathcal{Z}^d.$$

The lattices shown in Fig.2.1 and 2.2 are two typical examples with $Q = [0, 1), \mathcal{K} = \{1, 2\}$ and $Q = [0, 1)^2, \mathcal{K} = \{1\}$, respectively.

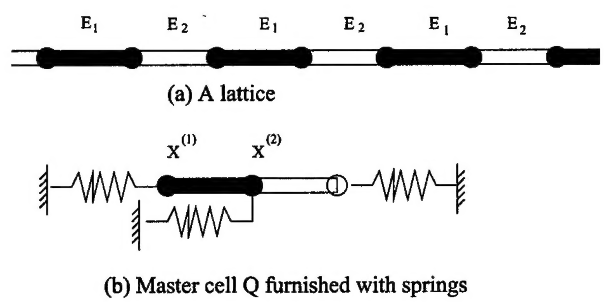


Fig. 2.1 A periodic lattice in $\mathbf{R}^1$

We now further specify the connectivity of lattices. By $\mathbf{b}^{(m,\kappa,n,\lambda)}$,we denote an elastic rod connecting the nodes $x^{(m,\kappa)}$ and $x^{(n,\lambda)}$ with intersect area A and length $b^{(m,\kappa,n,\lambda)}$. By $E^{(\kappa,n,\lambda)}$ we denote the Young's modules of the elastic rod $\mathbf{b}^{(\kappa,n,\lambda)}$. For the sake of simplicity, we will omit cell index m whenever $m = 0$. For instance, we write $x^{(0,\kappa)} = x^{(\kappa)}$, $\mathbf{b}^{(0,\kappa,n,\lambda)} = \mathbf{b}^{(\kappa,n,\lambda)}$, etc. We assume that

(C.1) Each node is connected to others by the rods,at least one node and at most M nodes;

(C.2) Any two nodes $x^{(m,\kappa)}$ and $x^{(n,\lambda)}$ are linked by the shortest chain $L_{m,\kappa,n,\lambda}$: $x^{(m,\kappa)} = x^{(n_1,\lambda_1)} \to x^{(n_2,\lambda_2)} \to x^{(n_3,\lambda_3)} \to \cdots \to x^{(n_s,\lambda_s)} = x^{(n,\lambda)}$ such that

$x^{(n_t,\lambda_t)}$ is connected to $x^{(n_{t+1},\lambda_{t+1})}, 1 \le t \le s-1$, and

$$|x^{(m,\kappa)} - x^{(n,\lambda)}| \le \sum_{1 \le t \le s-1} |x^{(n_{t+1},\lambda_{t+1})} - x^{(n_t,\lambda_t)}| \le \eta |x^{(m,\kappa)} - x^{(n,\lambda)}|,$$

where η is independent of m, n, κ and λ;

(C.3) The length of rods are uniformly bounded i.e. for any $x^{(m,\kappa)}$ and $x^{(n,\lambda)}$ which are connected, there holds

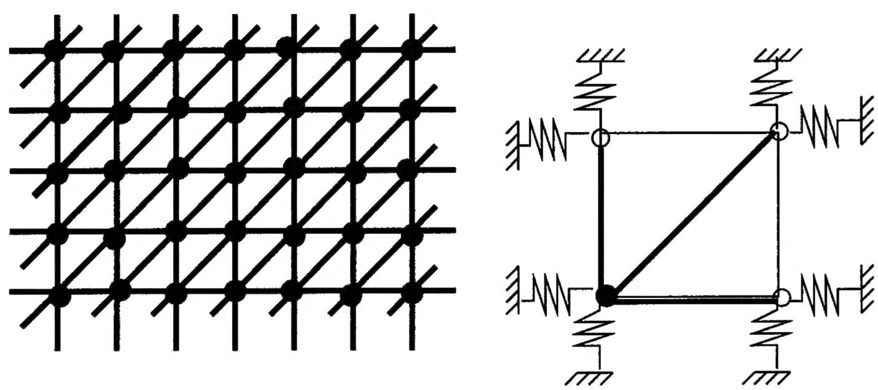
$$b_1 \le b^{(m,\kappa,n,\lambda)} = |x^{(m,\kappa)} - x^{(n,\lambda)}| \le b_2.$$

To effectively describe the connectivity of the lattice, we introduce B_κ and $B_{\kappa,\lambda}$

$$B_\kappa = \{(n,\lambda) \in \mathcal{Z}^d \times \mathcal{K} \text{ such that } x^{(0,\kappa)} \text{ and } x^{(n,\lambda)} \text{ are connected } \} \qquad (2.3a)$$

and

$$B_{\kappa,\lambda} = \{n \in \mathcal{Z}^d \text{ such that } (n,\lambda) \in B_\kappa\}. \qquad (2.3b)$$



(a) A lattice

(b) Master cell Q furnished with springs

Fig. 2.2 A periodic lattice in $\mathbf{R}^2$

B_κ and $B_{\kappa,\lambda}$, based on the connectivity of the nodes $x^{(0,\kappa)}$ in the cell Q_0, can be periodically generalized to sets $B_\kappa^{(m)}$ and $B_{\kappa,\lambda}^{(m)}$ for all $m \in \mathcal{Z}^d$ by the integer translation. Due to the periodicity , it is easy to verify that

$$n \in B_{\kappa,\lambda} \text{ if and only if } -n \in B_{\lambda,\kappa} \qquad (2.4)$$

and

$$E^{(\kappa,n,\lambda)} = E^{(\lambda,-n,\kappa)}. \qquad (2.5)$$

We further assume that lattices are rigid in the sense that if the new configurations of all cells resulted from a continuous transformation, which remains the lengthes of all rods fixed, are congruent to the original ones. A cell is rigid if the new configuration of the cell resulted from a continuous transformation, which remains the lengthes of all rods of the cell unchanged, are congruent to the original one. These difinitons of rigidity are based on the graph theory and coincide with those of [1]. Obviously, a lattice is rgid if all cells of the lattice

are rigid, but a rigid lattice may have some non-rigid cells. For instance, a cell consisting of triangles in two dimensions and tetrahedrons in three dimensions for which each edge is an elastic rod, is rigid, a lattice which is composed of such cells is referred as a triangular or tetrahedral lattice. Of course, a rigid lattice may not be triangular or tetrahedral. For the various definition of rigidity in general and verification of rigidity for a lattice or a truss system, which is not trivial, we refer to [2] and also to [1, 11, 27, 30].

A lattice is characterize by the local structure $\mathcal{K}$, the global and periodical translation on $\mathcal{Z}^d$, and the connectivity B_κ. We now denote the lattice with the above structures by $\mathcal{G} = \mathcal{G}(\mathcal{K}, \mathcal{Z}^d, B_\kappa)$.

2.2. A Truss Problem on Periodic Lattice

Let $u = (u_m)_{m\in\mathcal{Z}^d}$ and $u_m = (u_{m,\kappa})_{\kappa\in\mathcal{K}}$ be a grid functions on $\mathcal{G}$ and $\mathcal{K}$, respectively, and each $u_{m,\kappa}$ is a vector $(u^1_{m,\kappa}, u^2_{m,\kappa}, \dots, u^s_{m,\kappa})^\top$ with s-components. In one dimension, $s = d = 1$, and $u_{m,\kappa}$ denotes the displacement for elastic rods or the temperature of heat problems at the node $x^{(m,\kappa)}$. In two and three dimensions, $s = 1$ when $u_{m,\kappa}$ denotes the temperature. For the elastic problem, $s \geq d$. If the connections of rods are non-rigid, then $s = d$, and $u_{m,\kappa}$ denotes the displacement at the node $x^{(m,\kappa)}$. If the connection of rods are rigid, then $s = d(d+1)/2$ for $d = 2, 3$, and $u_{m,\kappa}$ denotes the displacement and rotation at the node $x^{(m,\kappa)}$. We furnish the rods with springs in the axis directions at each node with Hook's coefficients denoted by diagonal matrices $\mathbf{C}^{(m,\kappa)} = \mathbf{C}^{(\kappa)}, m \in \mathcal{Z}^d, \kappa \in \mathcal{K}$. We assume that the ratio of the length of rods and the intersect area A of rods >> 1. For the convenience to characterize the nature of our methodology, we will focus on the case that s=d , namely, the rods join with hinges, and the bending is not considered here.

If external forces exert on the rods at the nodes, denoted by $f = (f_m) = (f_{m,\kappa})_{(m,\kappa)\in\mathcal{Z}^d\times\mathcal{K}}$, we have the equilibrium equation

$$-\sum_{(n,\lambda)\in B_\kappa} \mathbf{E}^{(\kappa,n,\lambda)} \frac{(u_{m+n,\lambda} - u_{m,\kappa})}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|^2} + \mathbf{C}^{(\kappa)} u_{m,\kappa} = f_{m,\kappa}, \forall m \in \mathcal{Z}^d, \forall \kappa \in \mathcal{K}. \tag{2.6}$$

with

$$\mathbf{E}^{(\kappa,n,\lambda)} = AE^{(\kappa,n,\lambda)} \frac{(x^{(n,\lambda)} - x^{(\kappa)})}{|x^{(n,\lambda)} - x^{(\kappa)}|} \frac{(x^{(n,\lambda)} - x^{(\kappa)})^\top}{|x^{(n,\lambda)} - x^{(\kappa)}|} \tag{2.7a}$$

which is a matrix for $s = d > 1$ and a scalar quantity $AE^{(\kappa,n,\lambda)}$ for $s = 1, 1 \leq d \leq 3$, and

$$\mathbf{C}^{(\kappa)} = diag(C_1^{(\kappa)}, C_2^{(\kappa)}, \dots, C_d^{(\kappa)}), \quad C_\ell^{(\kappa)} \geq 0. \tag{2.7b}$$

Similarly, $\mathbf{C}^{(\kappa)}$ is a matrix for $s = d > 1$ and a scalar quantity $C^{(\kappa)}$ for $s = 1, 1 \leq d \leq 3$.

Let $H^1(\mathcal{G})$ and $L^2(\mathcal{G})$ be the Sobolev spaces over the lattice $\mathcal{G}$ with the norms

$$||u||^2_{L^2(\mathcal{G})} = \sum_{m\in\mathcal{Z}^d} \sum_{\kappa\in\mathcal{K}} |u_{m,\kappa}|^2 \tag{2.8}$$

and

$$\|u\|^2_{H^1(\mathcal{G})} = |u|^2_{H^1(\mathcal{G})} + \|u\|^2_{L^2(\mathcal{G})} \tag{2.9a}$$

where $|u|_{H^1(\mathcal{G})}$ is the semi-norm,

$$|u|^2_{H^1(\mathcal{G})} = \sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}}\sum_{(n,\lambda)\in B_\kappa} \frac{|u_{m+n,\lambda} - u_{m,\kappa}|^2}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|^2}. \tag{2.9b}$$

The corresponding variational problem is defined as

$$B(u,v) = F(v) \tag{2.10}$$

with the bilinear form on $H^1(\mathcal{G}) \times H^1(\mathcal{G})$

$$\begin{aligned} B(u,v) = \ & \textstyle\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}}\{\sum_{(n,\lambda)\in B_\kappa} \frac{1}{2}\langle \mathbf{E}^{(\kappa,n,\lambda)} \frac{(u_{m+n,\lambda} - u_{m,\kappa})}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|}, \\ & \frac{(v_{m+n,\lambda} - v_{m,\kappa})}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|}\rangle + \langle \mathbf{C}^{(\kappa)} u_{m,\kappa}, v_{m,\kappa}\rangle\} \end{aligned} \tag{2.11a}$$

and the linear functional on $H^1(\mathcal{G})$

$$F(v) = \sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}} \langle f_{m,\kappa}, v_{m,\kappa}\rangle, \tag{2.11b}$$

where $\langle x,y\rangle = \sum_{j=1}^d x_j y_j$ is the inner product of two vectors in $\mathbf{R}^d$, and $|x|^2 = \langle x,x\rangle$.

The energy of the elastic rods is

$$\begin{aligned} G(u) = B(u,u) = \ & \textstyle\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}}\{\sum_{(n,\lambda)\in B_\kappa} \frac{1}{2}\langle \mathbf{E}^{(\kappa,n,\lambda)} \frac{(u_{m+n,\lambda} - u_{m,\kappa})}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|}, \\ & \frac{(u_{m+n,\lambda} - u_{m,\kappa})}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|}\rangle + \langle \mathbf{C}^{(\kappa)} u_{m,\kappa}, u_{m,\kappa}\rangle\}. \end{aligned}$$

The energy space denoted by $E(\mathcal{G})$ is the family of all grid functions u on $\mathcal{G}$ with finite energy $G(u)$, and $\|u\|_{E(\mathcal{G})} = G(u)^{1/2}$ is referred as the energy norm of u.

Proposition 2.1 If a lattice $\mathcal{G}$ is rigid, then

$$\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}}\{\sum_{(n,\lambda)\in B_\kappa} \langle \mathbf{E}^{(\kappa,n,\lambda)} \frac{(u_{m+n,\lambda} - u_{m,\kappa})}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|}, \frac{(u_{m+n,\lambda} - u_{m,\kappa})}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|}\rangle = 0. \tag{2.12}$$

if and only if u is a rigid body motion.

Proof The proof is technical, we refer to [2].

Triangular and tetrahedral lattices are rigid, but a rigd lattice may not be triangular or tetrahedral. It can be proved that the lattice shown in Fig. 2.3 is rigid, but is not of triangular type. The lattice shown in Fig. 2.4 is neither rigid and nor of triangular type.

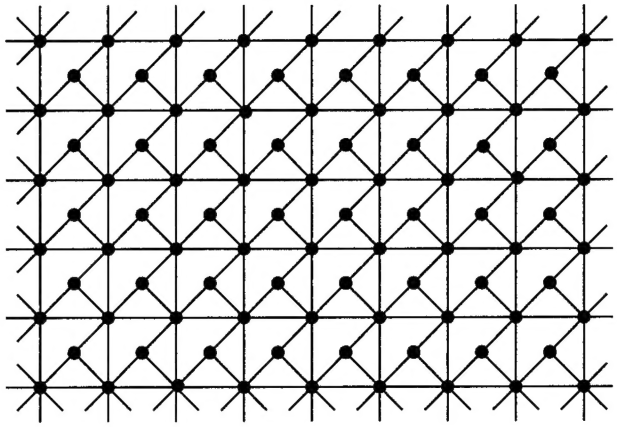
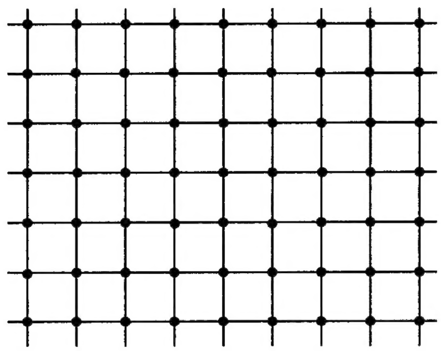
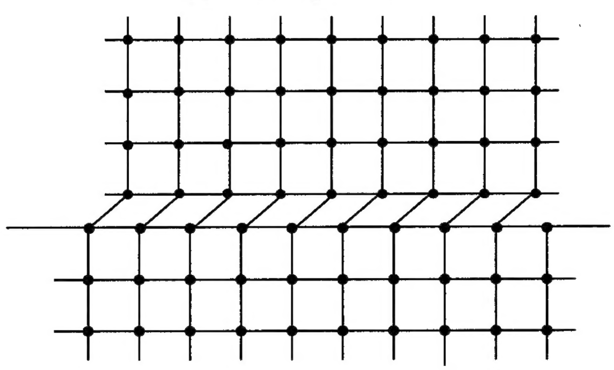


Fig. 2.3 A rigid and non-triangular lattice in $\mathbf{R}^2$



(a) A non-rigid lattice



(b) A non-rigid body motion on a non-rigid lattice

Fig. 2.4 A non-rigid lattice in $\mathbf{R}^2$

Lemma 2.2 The bilinear form B on $H^1(\mathcal{G}) \times H^1(\mathcal{G})$ given in (2.10) is continuous, and it is coercive if $\mathbf{C}^{(\kappa)} > 0$ for all $\kappa \in \mathcal{K}$.

Proof. Note that

$$\frac{\left(x^{(n,\lambda)} - x^{(\kappa)}\right)^\top}{\left|x^{(n,\lambda)} - x^{(\kappa)}\right|} \frac{\left(x^{(n,\lambda)} - x^{(\kappa)}\right)}{\left|x^{(n,\lambda)} - x^{(\kappa)}\right|} = (cos\theta_1, cos\theta_2, \ldots, cos\theta_d)^T (cos\theta_1, cos\theta_2, \ldots, cos\theta_d)$$

where θ_ℓ is the angle between the rod $\mathbf{b}^{(\kappa,n,\lambda)}$ and the x_ℓ-axis. We have immediately

$$|B(u,v)| \le C\|u\|_{H^1(\mathcal{G})}\|v\|_{H^1(\mathcal{G})}$$

with

$$C = Max\{\max_{n\in B_{\kappa,\lambda},\kappa,\lambda\in\mathcal{K}} AE^{(\kappa,n,\lambda)}, \max_{1\le\ell\le d,\kappa\in\mathcal{K}} C_\ell^{(\kappa)}\}.$$

Hence, the bilinear form B is continuous. Since $\mathbf{C}^{(\kappa)} > 0$ for all $\kappa \in \mathcal{K}$, we have

$$\langle \mathbf{C}^{(\kappa)} u_{m,\kappa}, u_{m,\kappa}\rangle \cong \|u\|^2_{L^2(\mathcal{G})}.$$

Here and hereafter "$\cong$" means equivalent with constants independent of major subjects, e.g. the functions u. For $s = 1$, there holds

$$\begin{aligned}&\langle \mathbf{E}^{(\kappa,n,\lambda)}\frac{(u_{m+n,\lambda}-u_{m,\kappa})}{|x^{(m+n,\lambda)}-x^{(m,\kappa)}|}, \frac{(u_{m+n,\lambda}-u_{m,\kappa})}{|x^{(m+n,\lambda)}-x^{(m,\kappa)}|}\rangle\\ &= \frac{AE^{(\kappa,n,\lambda)}}{|x^{(m+n,\lambda)}-x^{(m,\kappa)}|^4}|\langle x^{(m+n,\lambda)}-x^{(m,\kappa)}, u_{m+n,\lambda}-u_{m,\kappa}\rangle|^2\\ &= \frac{AE^{(\kappa,n,\lambda)}}{|x^{(m+n,\lambda)}-x^{(m,\kappa)}|^2}|u_{m+n,\lambda}-u_{m,\kappa}|^2 \ge d_1|u|^2_{H^1(\mathcal{G})}\end{aligned}$$

with $d_1 = \min_{n\in B_{\kappa,\lambda},\kappa,\lambda\in\mathcal{K}} AE^{(\kappa,n,\lambda)}$. For $s = d > 1$,

$$\begin{aligned}&\langle \mathbf{E}^{(\kappa,n,\lambda)}\frac{(u_{m+n,\lambda}-u_{m,\kappa})}{|x^{(m+n,\lambda)}-x^{(m,\kappa)}|}, \frac{(u_{m+n,\lambda}-u_{m,\kappa})}{|x^{(m+n,\lambda)}-x^{(m,\kappa)}|}\rangle\\ &= \frac{AE^{(\kappa,n,\lambda)}}{|x^{(m+n,\lambda)}-x^{(m,\kappa)}|^4}|\langle x^{(m+n,\lambda)}-x^{(m,\kappa)}, u_{m+n,\lambda}-u_{m,\kappa}\rangle|^2\\ &= \frac{AE^{(\kappa,n,\lambda)}}{|x^{(m+n,\lambda)}-x^{(m,\kappa)}|^2}|u_{m+n,\lambda}-u_{m,\kappa}|^2 cos^2\phi_{\kappa,n,\lambda} \ge d_1|u|^2_{H^1(\mathcal{G})}\end{aligned} \tag{2.13}$$

where $d_1 = \min_{\kappa\in\mathcal{K},(n,\lambda)\in B_\kappa} AE^{(\kappa,n,\lambda)}cos^2\phi_{\kappa,n,\lambda}$. By the rigidity assumption, u will be a rotation if $d_1 = 0$, which is not in $L^2(\mathcal{G})$. Hence, $d_1 > 0$ for $u \in H^1(\mathcal{G})$, and

$$B(u,u) \ge d_1|u|^2_{H^1(\mathcal{G})} + d_2\|u\|^2_{L^2(\mathcal{G})} \ge d\|u\|^2_{H^1(\mathcal{G})}$$

with $d_2 = \min_{1\le\ell\le d,\kappa\in\mathcal{K}} C_\ell^{(\kappa)} > 0$. and $d = min\{d_1, d_2\}$. □

Theorem 2.3 Suppose that $\mathbf{C}^{(\kappa)} \not\equiv 0$ for all $\kappa \in \mathcal{K}$. Then, for any $f \in (H^1(\mathcal{G}))^{-1}$, the variational equation (2.7) has a unique solution $u \in H^1(\mathcal{G})$, and

$$\|u\|_{H^1(\mathcal{G})} \le C\|f\|_{(H^1(\mathcal{G}))^{-1}}.$$

In particular, if $f \in L^2(\mathcal{G})$, there holds

$$\|u\|_{H^1(\mathcal{G})} \le C\|f\|_{L^2(\mathcal{G})}. \tag{2.14}$$

Proof. The theorem follows from the previous lemmas and Lax-Milgram Theorem. □

Remark 2.1 The condition that $\mathbf{C}^{(\kappa)} > 0$ for all $\kappa \in \mathcal{K}$ can be weakened to $C_\ell^{(\kappa)} \not\equiv 0$, the energy norm $\|u\|_{E(\mathcal{G})}$ is equivalent to the norm $\|u\|_{H^1(\mathcal{G})}$ due to the connectivity assumption [2]. If $C_\ell^{(\kappa)} \equiv 0$, Theorem 2.2 can not stand because the energy space $E(\mathcal{G})$ is not equivalent to $H^1(\mathcal{G})$ and it is not embedded in

$L^2(\mathcal{G})$. Because of the importance of lattice problems without absolute terms (i.e. $C_\ell^{(\kappa)} \equiv 0$) in applications we shall elaborate it in Section 3 and 4.

The relation between the solution of the equilibrium equation (2.6) and the solution of the variational equation (2.10) is given in next Theorem.

Theorem 2.4 If $u \in H^1(\mathcal{G})$ is the solution of the variational equation (2.10), then it satisfies the equilibrium equation (2.6). Vice versa, If $u \in H^1(\mathcal{G})$ solves the equilibrium equation (2.6), it satisfies the variational equation (2.10).

Proof. We first prove that the solution u of the equilibrium equation (2.6) satisfies the variation equation (2.10). For $v \in H^1(\mathcal{G})$, multiplying (2.6) with $v_{m,k}$ and summarizing with respect to m and κ, we have

$$\begin{aligned} &- \sum_{m\in\mathcal{Z}^d} \sum_{\kappa,\lambda\in\mathcal{K}} \sum_{n\in B_{\kappa,\lambda}} \langle \mathbf{E}^{(\kappa,n,\lambda)} \frac{u_{m+n,\lambda} - u_{m,\kappa}}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|}, \frac{v_{m,\kappa}}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|} \rangle \\ &+ \sum_{m\in\mathcal{Z}^d} \sum_{\kappa\in\mathcal{K}} \langle \mathbf{C}^{(\kappa)} u_{m,\kappa}, v_{m,\kappa} \rangle = \sum_{m\in\mathcal{Z}^d} \sum_{\kappa\in\mathcal{K}} \langle f_{m,\kappa}, v_{m,\kappa} \rangle, \end{aligned}$$

The above sums exist since $u, v \in H^1(\mathcal{G})$. Letting $m + n = m'$ and $n = -n'$, we get

$$\begin{aligned} &\sum_{m\in\mathcal{Z}^d} \sum_{\kappa,\lambda\in\mathcal{K}} \sum_{n\in B_{\kappa,\lambda}} \langle \mathbf{E}^{(\kappa,n,\lambda)} \frac{u_{m+n,\lambda} - u_{m,\kappa}}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|}, \frac{v_{m,\kappa}}{|x^{(m+n,\lambda)} - x^{(m,\kappa)}|} \rangle \\ = &\sum_{m'\in\mathcal{Z}^d} \sum_{\kappa,\lambda\in\mathcal{K}} \sum_{-n'\in B_{\kappa,\lambda}} \langle \mathbf{E}^{(\kappa,-n,\lambda)} \frac{u_{m',\lambda} - u_{m'+n',\kappa}}{|x^{(m',\lambda)} - x^{(m'+n',\kappa)}|}, \frac{v_{m'+n',\kappa}}{|x^{(m',\lambda)} - x^{(m'+n',\kappa)}|} \rangle \end{aligned}$$

Due to the properties (2.4) and (2.5) :

$$\begin{aligned} \mathbf{E}^{(\kappa,-n',\lambda)} &= \mathbf{E}^{(\lambda,n',\kappa)} \\ -n' \in B_{\kappa,\lambda} \quad &iff \quad n' \in B_{\lambda,\kappa} \end{aligned}$$

there hold

$$\begin{aligned} &\sum_{m'\in\mathcal{Z}^d} \sum_{\kappa,\lambda\in\mathcal{K}} \sum_{-n'\in B_{\kappa,\lambda}} \langle \mathbf{E}^{(\kappa,-n',\lambda)} \frac{u_{m',\lambda} - u_{m'+n',\kappa}}{|x^{(m',\lambda)} - x^{(m'+n',\kappa)}|}, \frac{v_{m'+n',\kappa}}{|x^{(m',\lambda)} - x^{(m'+n',\kappa)}|} \rangle \\ = -&\sum_{m'\in\mathcal{Z}^d} \sum_{\kappa,\lambda\in\mathcal{K}} \sum_{n'\in B_{\lambda,\kappa}} \langle \mathbf{E}^{(\lambda,n',\kappa)} \frac{u_{m',\lambda} - u_{m'+n',\kappa}}{|x^{(m',\lambda)} - x^{(m'+n',\kappa)}|}, \frac{v_{m'+n',\kappa}}{|x^{(m',\lambda)} - x^{(m'+n',\kappa)}|} \rangle \end{aligned} \tag{2.15}$$

Which leads to the (2.10) immediately.

We now show that the variational solution $u \in H^1(\mathcal{G})$ solves the equilibrium equation (2.10). Let $v \in H^1(\mathcal{G})$ be such that $v_{m,\kappa} = 0$ for all $\kappa \in \mathcal{K}$ except

$\kappa = 1$. Then the variational equation leads to

$$\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}}\sum_{n\in B_{\kappa,1}} \langle \frac{1}{2}\mathbf{E}^{(\kappa,n,1)} \frac{u_{m+n,1}-u_{m,\kappa}}{|x^{(m+n,1)}-x^{(m,\kappa)}|}, \frac{v_{m+n,1}}{|x^{(m+n,1)}-x^{(m,\kappa)}|}\rangle$$

$$- \sum_{m\in\mathcal{Z}^d}\sum_{\lambda\in\mathcal{K}}\sum_{n\in B_{1,\lambda}} \langle \frac{1}{2}\mathbf{E}^{(1,n,1)} \frac{u_{m+n,\lambda}-u_{m,1}}{|x^{(m+n,\lambda)}-x^{(m,1)}|}, \frac{v_{m,1}}{|x^{(m+n,\lambda)}-x^{(m,1)}|}\rangle$$

$$+ \sum_{m\in\mathcal{Z}^d} \langle \mathbf{C}^{(\kappa)} u_{m,1},, v_{m,1}\rangle = \sum_{m\in\mathcal{Z}^d}\langle f_{m,1}, v_{m,1}\rangle$$

Selecting $v_{m,\kappa}$ such that $v_{m,\kappa} = 0$ for all $\kappa \in \mathcal{K}, m \in \mathcal{Z}$ except $v_{\tilde{m},1}$, Due to (2.15), we obtain

$$-\sum_{\lambda\in B_\kappa}\sum_{n\in B_{\tilde{m},\lambda}} \langle \mathbf{E}^{(\kappa,n,\lambda)} \frac{u_{\tilde{m}+n,\lambda}-u_{\tilde{m},1}}{|x^{(\tilde{m}+n,\lambda)}-x^{(\tilde{m},1)}|}, \frac{v_{1.1}}{|x^{(\tilde{m}+n,\lambda)}-x^{(\tilde{m},1)}|}\rangle$$

$$+ \quad \langle \mathbf{C}^{(1)} u_{\tilde{m},1}, v_{\tilde{m},1}\rangle = \langle f_{\tilde{m},1}, v_{\tilde{m},1}\rangle$$

which implies

$$-\sum_{\lambda\in B_\kappa}\sum_{n\in B_{\tilde{m},\lambda}} \mathbf{E}^{(\kappa,n,\lambda)} \frac{u_{\tilde{m}+n,\lambda}-u_{\tilde{m},1}}{|x^{(\tilde{m}+n,\lambda)}-x^{(\tilde{m},1)}|^2} + \mathbf{C}^{(1)} u_{\tilde{m},1} = f_{\tilde{m},1}$$

Similarly, there holds for any $m \in \mathcal{Z}^d$

$$-\sum_{\lambda\in B_\kappa}\sum_{n\in B_{1,\lambda}} \mathbf{E}^{(\kappa,n,\lambda)} \frac{u_{m+n,\lambda}-u_{m,1}}{|x^{(m+n,\lambda)}-x^{(m,1)}|^2} + \mathbf{C}^{(1)} u_{m,1} = f_{m,1},$$

Actually, the above argument can be carried for any $\kappa \in \mathcal{K}$. Thus, we have the equation (2.6). □

2.3 Fourier transform for lattice problems

For grid functions u on the periodic lattice $\mathcal{G}$ in entire spaces, we introduce the Fourier transform

$$\mathcal{F}(u) = \sum_{m\in\mathcal{Z}^d} u_m e^{i\langle m,t\rangle} = \hat{u}(t) \tag{2.16a}$$

which is a linear functional over the space $C^\infty_{per}(I^d) = \{\hat{u} \in C^\infty(I^d) | \hat{u}(t) \text{ is a } 2\pi\text{-periodic function }\}$, where $I^d = (-\pi,\pi)^d$. $\hat{u}(t)$ is a complex-valued vector function, $(\hat{u}_\kappa(t))_{\kappa\in\mathcal{K}}$, and each $\hat{u}_\kappa(t)$ has s components $\hat{u}^\ell_\kappa(t), 1 \le \ell \le s$, and

$$\hat{u}_\kappa(t) = \sum_{m\in\mathcal{Z}^d} u_{m,\kappa} e^{i<m,t>}, \quad \forall\kappa \in \mathcal{K}. \tag{2.16b}$$

The inverse Fourier transform is defined as

$$\mathcal{F}^{-1}(\hat{u}) = (u_m)_{m\in\mathcal{Z}^d} \tag{2.17a}$$

for any $\hat{u}(t) \in C^\infty_{per}(I^d)$, and

$$u_m = (2\pi)^{-d}\int_{I^d} \hat{u}(t) e^{-i<m,t>} dt \tag{2.17b}$$

For lattice problems we are interested in some of specific spaces over $\mathcal{G}$, e.g. $L^2(\mathcal{G})$, then $\mathcal{F}(u) \in L^2(I^d)$, which has a stronger topology than the linear functional on the space $C^\infty_{per}(I^d)$. In particular, we are interested in the Fourier transform on the spaces $L^2_\nu(\mathcal{G}), \nu \geq 0$ with the norm

$$\|u\|^2_{L^2_\nu(\mathcal{G})} = \sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}}(1+|m|^2)^\nu|u_{m,\kappa}|^2.$$

Lemma 2.5 The Fourier transform realizes isomorphism: $L^2(\mathcal{G}) \leftrightarrow L^2(I^d)$ and $L^2_\nu(\mathcal{G}) \leftrightarrow H^\nu_{per}(I^d)$, where $H^\nu_{per}(I^d)$ is the subspace of 2π- periodic functions in $H^\nu(I^d)$, and

$$\|u\|^2_{L^2(\mathcal{G})} = (2\pi)^{-d}\|\hat{u}\|^2_{L^2(I^d)}, \tag{2.18a}$$

$$\|u\|^2_{L^2_\nu(\mathcal{G})} \cong \|\hat{u}\|^2_{H^\nu(I^d)}. \tag{2.18b}$$

Proof. It is easy to verify that

$$\begin{aligned}\|\hat{u}\|^2_{L^2(I^d)} &= \int_{I^d}\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}}\sum_{0\leq\ell\leq s}|\hat{u}^\ell_\kappa|^2dt = \int_{I^d}\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}}\sum_{0\leq\ell\leq s}|u^\ell_{m,\kappa}e^{i<m,t>}|^2dt\\ &= (2\pi)^d\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}}\sum_{0\leq\ell\leq s}|u^\ell_{m,\kappa}|^2 = (2\pi)^d\|u\|^2_{L^2(\mathcal{G})}\end{aligned}$$

which implies an isomorphism : $L^2(\mathcal{G}) \leftrightarrow L^2(I^d)$ and (2.18a). For $\hat{u}(t) \in H^\nu_{per}(I^d)$ with integer $\nu \geq 0$, there holds

$$D^\alpha\hat{u}(t) = \prod_{\ell=1}^{d}(im_\ell)^{\alpha_\ell}\hat{u}(t)$$

for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ with $|\alpha| = \sum_{1\leq i\leq d}\alpha_i \leq \nu$, which leads to

$$\|\hat{u}\|^2_{H^\nu(I^d)} \cong \sum_{m\in\mathcal{Z}^d}(1+|m|^2)^\nu|u_m|^2 = \|u\|^2_{L^2_\nu(\mathcal{G})}.$$

For non-integer $\nu > 0$, $H^\nu_{per}(I^d)$ is defined as an interpolation space, and (2.18b) stands for non-integer ν as well. □

We now apply Fourier transform to the variational problem (2.10). We introduce a bilinear form $\hat{B}$ and a linear functional $\hat{F}$, namely,

$$\begin{aligned}\hat{B}\left(\hat{u},\hat{v}\right) &= \int_{I^d}\sum_{\kappa\in\mathcal{K}}\{\sum_{(n,\lambda)\in B_\kappa}\frac{1}{2}\langle\mathbf{E}^{(\kappa,n,\lambda)}\frac{(\hat{u}_\lambda e^{-i<n,t>}-\hat{u}_\kappa)}{|x^{(n,\lambda)}-x^{(\kappa)}|},\\ &\quad\frac{(\hat{v}_\lambda e^{-i<n,t>}-\hat{v}_\kappa)}{|x^{(n,\lambda)}-x^{(\kappa)}|}\rangle + \langle\mathbf{C}^{(\kappa)}\hat{u}_\kappa,\hat{v}_\kappa\rangle\}dt\end{aligned} \tag{2.19}$$

and

$$\hat{F}(\hat{v}) = \sum_{\kappa\in\mathcal{K}}\int_{I^d}\langle\hat{f},\hat{v}_\kappa\rangle dt. \tag{2.20}$$

Then we have the following lemma.

Lemma 2.6 Let $u, v \in H^1(\mathcal{G})$ and $f \in L^2(\mathcal{G})$, and let $\hat{u}, \hat{v}, \hat{f}$ be their Fourier transform, respectively. Then there hold

$$B(u,v) = (2\pi)^{-d}\hat{B}(\hat{u},\hat{v}) \tag{2.21a}$$

and

$$F(v) = (2\pi)^{-d}\hat{F}(\hat{v}). \tag{2.21b}$$

Proof. For $\hat{v} = \sum_{m\in\mathcal{Z}^d} v_m e^{i<m,t>}$ and $\hat{f} = \sum_{m\in\mathcal{Z}^d} f_m e^{i<m,t>}$, there holds

$$\begin{aligned}\hat{F}(\hat{v}) &= \sum_{\kappa\in\mathcal{K}}\int_{I^d}\langle \hat{f}_\kappa, \hat{v}_\kappa\rangle dt = \sum_{\kappa\in\mathcal{K}}\int_{I^d}\sum_{n,m\in\mathcal{Z}^d}\langle f_{n,\kappa}, v_{m,\kappa}\rangle e^{i<(n-m),t>}dt \\ &= (2\pi)^d\sum_{\kappa\in\mathcal{K}}\sum_{m\in\mathcal{Z}^d} < f_{m,\kappa}, v_{m,\kappa} > .\end{aligned}$$

This leads to (2.21). Similarly we have

$$\sum_{\kappa\in\mathcal{K}}\int_{I^d}\langle \mathbf{C}^{(\kappa)}\hat{u}_\kappa, \hat{v}_\kappa\rangle dt = (2\pi)^d\sum_{\kappa\in\mathcal{K}}\sum_{m\in\mathcal{Z}^d}\langle \mathbf{C}^{(\kappa)}u_{m,\kappa}, v_{m,\kappa}\rangle. \tag{2.22}$$

It is easy to see that

$$\begin{aligned}&\int_{I^d}\sum_{\kappa\in\mathcal{K}}\sum_{(n,\lambda)\in B_\kappa}\frac{1}{2}\langle \mathbf{E}^{(\kappa,n,\lambda)}\frac{(\hat{u}_\lambda e^{-i<n,t>}-\hat{u}_\kappa)}{|x^{(n,\lambda)}-x^{(\kappa)}|}, \frac{(\hat{v}_\lambda e^{-i<n,t>}-\hat{v}_\kappa)}{|x^{(n,\lambda)}-x^{(\kappa)}|}\rangle dt \\ =\ &\int_{I^d}\sum_{\kappa\in\mathcal{K}}\sum_{(n,\lambda)\in B_\kappa}\frac{1}{2}\langle \mathbf{E}^{(\kappa,n,\lambda)}\sum_{m\in\mathcal{Z}^d}\frac{(u_{m,\lambda}e^{i<(m-n),t>}-u_{m,\kappa}e^{i<m,t>})}{|x^{(n,\lambda)}-x^{(\kappa)}|}, \\ &\sum_{m'\in\mathcal{Z}^d}\frac{(v_{m',\lambda}e^{-i<m'-n,t>}-v_{m',\kappa}e^{i<m',t>})}{|x^{(n,\lambda)}-x^{(\kappa)}|}\rangle dt \\ =\ &\int_{I^d}\sum_{\kappa\in\mathcal{K}}\sum_{(n,\lambda)\in B_\kappa}\sum_{m\in\mathcal{Z}^d}\frac{1}{2}\langle \mathbf{E}^{(\kappa,n,\lambda)}\frac{(u_{m+n,\lambda}-u_{m,\kappa})}{|x^{(n,\lambda)}-x^{(\kappa)}|}, \frac{(v_{m+n,\lambda}-v_{m,\kappa})}{|x^{(n,\lambda)}-x^{(\kappa)}|}\rangle dt \\ =\ &(2\pi)^d\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}}\sum_{(n,\lambda)\in B_\kappa}\frac{1}{2}\langle \mathbf{E}^{(\kappa,n,\lambda)}\frac{(u_{m+n,\lambda}-u_{m,\kappa})}{|x^{(m+n,\lambda)}-x^{(m,\kappa)}|}, \frac{(v_{m+n,\lambda}-v_{m,\kappa})}{|x^{(m+n,\lambda)}-x^{(m,\kappa)}|}\rangle.\end{aligned}$$

which together with (2.22) yields (2.21). □

In order to properly define a variational problem over $I^d \times K_Q$,we need to introduce new function spaces. Let $L^2(K_Q)$ and $H^1(K_Q)$ be the spaces of grid functions on K_Q with the following norms

$$\|w\|^2_{L^2(K_Q)} = \sum_{\kappa\in\mathcal{K}}|w_\kappa|^2 = \sum_{\kappa\in\mathcal{K}}\sum_{1\le l\le s}|w^l_\kappa|^2$$

and

$$\|w\|^2_{H^1(K_Q)} = \sum_{\kappa\in\mathcal{K}}\sum_{(n,\lambda)\in B_\kappa}\frac{|w_{n,\lambda}-w_\kappa|^2}{|x^{(n,\lambda)}-x^{(\kappa)}|^2} + \|w\|^2_{L^2(K_Q)}.$$

$L^2\left(I^d, H^1(K_Q)\right)$ and $L^2\left(I^d, L^2(K_Q)\right)$ are spaces furnished with the norms :

$$\|\hat{w}\|^2_{L^2\left(I^d,H^1(K_Q)\right)} = \int_{I^d}\|\hat{w}(t)\|^2_{H^1(K_Q)}dt$$

and

$$\|\hat{w}\|^2_{L^2\left(I^d,L^2(K_Q)\right)} = \int_{I^d} \|\hat{w}(t)\|^2_{L^2(K_Q)}dt.$$

Lemma 2.7 If $f \in H^l(\mathcal{G})$, then $\hat{f}(t) \in L^2\left(I^d, H^l(K_Q)\right), l = 0, 1$, and

$$\|\hat{f}(t)\|_{L^2\left(I^d,H^l(K_Q)\right)} = (2\pi)^d \|f\|_{H^l(\mathcal{G})}$$

Proof. The assertion follows easily from the definition of the spaces. □

Remark 2.2 The functions $\hat{u} = \hat{u}(t)$ in the space $H^l(I^d), l = 0, 1$ are vector functions with sq components, and the functions $\hat{u} = \hat{u}(t, x^{(\kappa)})$ in $L^2\left(I^d, L^2(K_Q)\right)$ are those defined on a semi-discrete domain $I^d \times K_Q$ with s components. Obviously, the space $L^2\left(I^d\right)$ coincides with the space $L^2\left(I^d, L^2(K_Q)\right)$, and

$$\|\hat{f}\|_{L^2\left(I^d,L^2(K_Q)\right)} = \|\hat{f}\|_{L^2(I^d)}.$$

But the space $H^1(I^d)$ is totally different from the space $L^2\left(I^d, H^1(K_Q)\right)$. The latter is related to the connectivity B_κ, and the former is not. Furthermore, the space $H^1(I^d)$ is an isomorphism of the space $L^2_1(\mathcal{G})$, and the space $L^2\left(I^d, H^1(K_Q)\right)$ is an isomorphism of the space $H^1(\mathcal{G})$ according to Lemma 2.6.

The bilinear form $\hat{B}$ in (2.20) and linear functional $\hat{F}$ in (2.21) are defined on $L^2\left(I^d, H^1(K_Q)\right) \times L^2\left(I^d, H^1(K_Q)\right)$ and $L^2\left(I^d, H^1(K_Q)\right)$, respectively. The energy space $\hat{E} = \hat{E}\left(I^d \times K_Q\right)$ is defined as one equivalent to $L^2\left(I^d, H^1(K_Q)\right)$ if $\mathbf{C}^{(\kappa)} \not\equiv 0$, with an energy norm

$$\|\hat{w}\|^2_{\hat{E}(I^d \times K_Q)} = \hat{B}(\hat{w}, \hat{w})^{1/2}.$$

We are now able to precisely address the variational problem over the domain $I^d \times K_Q$.

Theorem 2.8 Let $\hat{B}$ and $\hat{F}$ be the bilinear form and linear functional on $L^2\left(I^d, H^1(K_Q)\right) \times L^2\left(I^d, H^1(K_Q)\right)$ and $L^2\left(I^d, H^1(K_Q)\right)$, given in (2.19) and (2.20), respectively. If $\hat{f} \in L^2\left(I^d, L^2(K_Q)\right)$ and $\mathbf{C}^{(\kappa)} \not\equiv 0, \forall \kappa \in \mathcal{K}$, then the variational problem

$$\hat{B}\left(\hat{u}, \hat{v}\right) = \hat{F}(\hat{v}), \ \forall \hat{v} \in L^2\left(I^d, H^1(K_Q)\right) \tag{2.24}$$

has a unique solution $\hat{u} \in L^2\left(I^d, H^1(K_Q)\right)$, and

$$\|\hat{u}\|_{L^2\left(I^d,H^1(K_Q)\right)} \le C\|\hat{f}\|_{\left(L^2\left(I^d,L^2(K_Q)\right)\right)}. \tag{2.25}$$

Remark 2.3 Combining Theorem 2.8 with Theorem 2.3 and Lemma 2.6, we have that the equivalence between the problem (2.10) and the problem (2.24), i.e. the equation (2.24) has a unique solution $\hat{u} \in L^2\left(I^d, H^1(K_Q)\right)$ and the estimate (2.25) holds for $\hat{f} \in \left(I^d, L^2(K_Q)\right)$ if and only if the problem (2.10) has

a unique solution $u \in H^1(\mathcal{G})$ and the estimate (2.14) holds for $f = \mathcal{F}^{-1}(\hat{f}) \in L^2(\mathcal{G})$, and $u = \mathcal{F}^{-1}(\hat{u})$.

Applying Fourier transform to the equilibrium Equations (2.6) leads to an equilibrium equations over $I^d \times K_Q$

$$-\sum_{(n,\lambda)\in B_\kappa} \mathbf{E}^{(\kappa,n,\lambda)} \frac{(\hat{u}_\lambda e^{-i<n,t>} - \hat{u}_\kappa)}{|x^{(n,\lambda)} - x^{(\kappa)}|^2} + \mathbf{C}^{(\kappa)}\hat{u}_\kappa = \hat{f}_\kappa, \forall \kappa \in \mathcal{K}. \tag{2.26}$$

Then we have a theorem indicating the relation between the solution of (2.24) and the solution of (2.26).

Theorem 2.9 If $\hat{u} \in L^2\left(I^d, H^1(K_Q)\right)$ is the solution of the variational equation (2.24) with $\hat{f} \in L^2\left(I^d, L^2(K_Q)\right)$, then it satisfies the equilibrium equation (2.26). Vice versa, if $\hat{u} \in L^2\left(I^d, H^1(K_Q)\right)$ solves the equilibrium equation (2.26) with $\hat{f} \in L^2\left(I^d, L^2(K_Q)\right)$, then it satisfies the variational equation (2.24).

Proof. The proof is analogous to that for Theorem 2.3. □

2.4 Presentation Formula of Solutions

The equation (2.26) gives a system of linear equations

$$\boldsymbol{\sigma}(t)\hat{u}(t) = \hat{f}(t) \tag{2.27}$$

where $\hat{u} = \left(\hat{u}_1^T, \hat{u}_2^T \ldots \hat{u}_q^T\right)^T$, and $\hat{u}_\kappa = \left(\hat{u}_1^1, \ldots \hat{u}_\kappa^s\right)^T, \kappa \in \mathcal{K}$. $\hat{u}$ and $\hat{f}$ are vectors with sq components, and $\boldsymbol{\sigma}$ is a block matrix

$$\boldsymbol{\sigma} = \left(\boldsymbol{\sigma}_{\kappa,\lambda}\right)_{1\le\kappa,\lambda\le q}.$$

Each of block $\boldsymbol{\sigma}_{\kappa\lambda}$ is a $s \times s$ matrix. It follows from (2.26) that

$$\boldsymbol{\sigma}_{\kappa,\kappa} = \sum_{n\in B_{\kappa\kappa}} \left(1 - e^{-i<n,t>}\right) \mathbf{E}^{(\kappa,n,\kappa)} + \mathbf{C}^{(\kappa)}, \tag{2.28a}$$

$$\boldsymbol{\sigma}_{\kappa,\lambda} = -\sum_{n\in B_{\kappa\lambda}} e^{-i<n,t>}\mathbf{E}^{(\kappa,n,\lambda)}. \tag{2.28b}$$

Lemma 2.10 The matrix $\boldsymbol{\sigma}$ has following properties :

(1) $\boldsymbol{\sigma}$ is a Hermitian matrix and Hermitian block matrix;

(2) $\boldsymbol{\sigma}_{\kappa\lambda}(-t) = \boldsymbol{\sigma}_{\lambda,\kappa}(t)^T$ for $\lambda, \kappa \in \mathcal{K}$, and $\boldsymbol{\sigma}(-t) = \boldsymbol{\sigma}(t)^T$;

(3) $\boldsymbol{\sigma}(t)$ is a positive definite matrix for all $t \in I^d$ If $\mathbf{C}^{(\kappa)} \not\equiv 0$ for $\kappa \in \mathcal{K}$;

(4) If $\mathbf{C}^{(\kappa)} \equiv 0$, $\boldsymbol{\sigma}(t)$ is a semi-positive definite matrix for all $t \in I^d$ and is a positive definite matrix for all $t \in I^d$ for all $t \in I^d$ except $t = 0$.

Proof.

(1)It follows from (2.28b) and (2.4) – (2.5) that for $\kappa \neq \lambda$

$$\begin{aligned}\boldsymbol{\sigma}_{\kappa,\lambda} &= -\sum_{n\in B_{\kappa\lambda}} e^{-i<n,t>}\mathbf{E}^{(\kappa,n,\lambda)} = -\sum_{n\in B_{\lambda,\kappa}} e^{i<-n,t>}\mathbf{E}^{(\kappa,n,\lambda)} \\ &= -\sum_{m\in B_{\lambda,\kappa}} e^{i<m,t>}\mathbf{E}^{(\kappa,-m,\lambda)} = -\sum_{m\in B_{\lambda,\kappa}} e^{i<m,t>}\mathbf{E}^{(\lambda,m,\kappa)} = \overline{\boldsymbol{\sigma}}_{\lambda,\kappa}.\end{aligned}$$

For $\kappa = \lambda$, we shall write

$$B_{\kappa,\kappa} = B^-_{\kappa,\kappa} \cup B^+_{\kappa,\kappa}$$

where

$$B^+_{\kappa\kappa} = \{n = (n_1, n_2 \dots n_d) \in B_{\kappa\kappa} \mid n_l \geq 0, 1 \leq l \leq d\}$$
$$B^-_{\kappa\kappa} = \{n = (n_1, n_2 \dots n_d) \in B_{\kappa\kappa} \mid n_l \leq 0, 1 \leq l \leq d\}$$

Due to the definition (2.3) of $B_{\kappa,\lambda}$, $0 \notin B_{\kappa,\kappa}$, which implies $R^+_{\kappa,\kappa} \cap R^-_{\kappa,\kappa} = \emptyset$. By the property (2.4), $n \in B^+_{\kappa,\kappa}$ if and only if $-n \in B^-_{\kappa,\kappa}$. Therefore

$$\boldsymbol{\sigma}_{\kappa,\kappa} = \mathbf{C}^{(\kappa)} + 4\sum_{n\in B^+_{\kappa,\kappa}} \sin^2\frac{<n,t>}{2}\mathbf{E}^{(\kappa,n,\kappa)} \tag{2.29}$$

$\boldsymbol{\sigma}_{\kappa,\kappa}$ is a real matrix. Thus we have shown that $\boldsymbol{\sigma}$ is Hermitian block matrix. Note that $\mathbf{C}^{(\kappa)}$ and $\mathbf{E}^{(\kappa,n,\lambda)}$ are symmetric matrices, which implies the $\boldsymbol{\sigma}$ is a Hermitian matrix as well.

(2) For $\lambda \neq \kappa$, by the properties (2.4) and

$$\begin{aligned}\boldsymbol{\sigma}_{\kappa,\lambda}(-t) &= -\textstyle\sum_{n\in B_{\kappa,\lambda}} e^{-i<n,-t>}\mathbf{E}^{(\kappa,n,\lambda)} = -\sum_{-n\in B_{\lambda,\kappa}} e^{-i<-n,t>}\mathbf{E}^{(\kappa,n,\lambda)} \\ &= -\textstyle\sum_{m\in B_{\lambda,\kappa}} e^{-i<m,t>}\mathbf{E}^{(\kappa,-m,\lambda)} = -\sum_{m\in B_{\lambda,\kappa}} e^{-i<m,t>}\mathbf{E}^{(\lambda,m,\kappa)} = \boldsymbol{\sigma}_{\lambda,\kappa}(t).\end{aligned}$$

It is trivial by (2.29) that $\boldsymbol{\sigma}_{\kappa,\kappa}(t) = \boldsymbol{\sigma}_{\kappa,\kappa}(-t), \forall\kappa \in \mathcal{K}$. Since each block $\boldsymbol{\sigma}_{\lambda,\kappa}(t)$ is symmetric, we have $\boldsymbol{\sigma}(-t) = \boldsymbol{\sigma}(t)^\top$.

(3) Let $\hat{b}(t) = \left(\hat{b}(t)\right)_{\kappa\in\mathcal{K}} \in L^2(I^d, H^1(K_Q)$, and let $b \in \mathcal{F}^{-1}\left(\hat{b}(t)\right)$. Then, by Lemma 2.6, $b \in H^1(\mathcal{G})$, and

$$\int_{I^d}\langle\boldsymbol{\sigma}(t)\hat{b}(t), \hat{b}(t)\rangle dt = B(b, b)$$

$$= \sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in K_Q}\sum_{(n,\lambda)\in B_\kappa}\langle\mathbf{E}^{(\kappa,n,\lambda)}\left(b_{m+n,\lambda} - b_{m,\kappa}\right), \left(b_{m+n,\lambda} - b_{m,\kappa}\right)\rangle$$

$$+\sum_{\kappa\in\mathcal{K}}\langle\mathbf{C}^{(\kappa)}b_{m+\kappa}, b_{m+\kappa}\rangle \geq d_1|b|^2_{H^1(\mathcal{G})} + d_2||b||^2_{L^2(\mathcal{G})}$$

where d_1 and d_2 are given in (2.13). Note that

$$||b|^2_{H^1(\mathcal{G})} = \int_{I^d}|\hat{b}(t)|^2_{H^1(K_Q)}dt,\ ||b||^2_{L^2(\mathcal{G})} = \int_{I^d}||\hat{b}(t)||^2_{L^2(K_Q)}dt,$$

which implies that

$$\int_{I^d}\langle\boldsymbol{\sigma}(t)\hat{b}(t), \hat{b}(t)\rangle dt \geq d_1\int_{I^d}|\hat{b}(t)|^2_{H^1(K_Q)}dt + d_2\int_{I^d}||\hat{b}(t)||^2_{L^2(K_Q)}dt.$$

and that for almost every $t \in I^d$

$$\langle \sigma(t)\hat{b}(t), \hat{b}(t)\rangle \geq d_1 |\hat{b}(t)|^2_{H^1(K_Q)} + d_2 \|\hat{b}(t)\|^2_{L^2(K_Q)} \geq d_2 \|\hat{b}(t)\|^2_{L^2(K_Q)}.$$

with $d_2 > 0$. Note that $\boldsymbol{\sigma}(t)$ is a Hermitian matrix and analytic in t. Hence $\boldsymbol{\sigma}(t)$ is positive definite for all $t \in I^d$.

If $\mathbf{C}^{(\kappa)} \equiv 0$ for all $\kappa \in \mathcal{K}$, there holds

$$\langle \boldsymbol{\sigma}(t) b(t), \hat{b}(t)\rangle \geq d_1 |\hat{b}(t)|^2_{H^1(K_Q)} \geq 0.$$

$\boldsymbol{\sigma}(t)$ is semi-positive definite for all $t \in I^d$. By [?], $\det(\boldsymbol{\sigma}(t)) = 0$ if and only if $t = 0$, which implies that $\boldsymbol{\sigma}(t)$ is positive definite if $t \neq 0$. □

In terms of $\boldsymbol{\sigma}^{-1}(t)$,we now can derive a representation formula of the solution the lattice problem.

Theorem 2.11 If $\mathbf{C}^{(\kappa)} \not\equiv 0$ for $\kappa \in \hat{\mathcal{K}}$ and $f \in L^2(\mathcal{G})$, the solution of the lattice problem (2.6) can be represented by

$$u = \mathcal{F}^{-1}\left(\boldsymbol{\sigma}^{-1}(t)\hat{f}(t)\right) \tag{2.30a}$$

where $\hat{f} = \mathcal{F}(f) \in L^2(I^d)$, with

$$u_m = (2\pi)^{-d} \int_{I^d} \boldsymbol{\sigma}^{-1}(t)\hat{f}(t) e^{-i<m,t>} dt \tag{2.30b}$$

and

$$\|\mathcal{F}^{-1}\left(\boldsymbol{\sigma}(t)^{-1}\hat{f}(t)\right)\|_{H^1(\mathcal{G})} \leq C \|\hat{f}\|_{L^2(I^d, L^2(K_Q))}.$$

Proof. Since $\boldsymbol{\sigma}(t)$ is positive definite for all $t \in I^d$ if $\mathbf{C}^{(\kappa)} \not\equiv 0$,

$$\hat{u} = \boldsymbol{\sigma}^{-1}(t)\hat{f}(t)$$

is the solution of the equation (2.26). By Theorem 2.8, it solves the variational equation (2.24), and

$$\|\hat{u}\|_{L^2\left(I^d, H^1(K_Q)\right)} \leq C \|\hat{f}\|_{L^2\left(I^d, L^2(K_Q)\right)}.$$

Let $u = \mathcal{F}^{-1}(\hat{u}(t))$. Then $u \in H^1(\mathcal{G})$ and solves the equation (2.6), and

$$\begin{aligned} \|\mathcal{F}^{-1}\left(\boldsymbol{\sigma}(t)^{-1}\hat{f}(t)\right)\|_{H^1(\mathcal{G})} &= \|u\|_{H^1(\mathcal{G})} \leq C\|\hat{u}\|_{L^2\left(I^d, H^1(K_Q)\right)} \\ &\leq C\|\hat{f}\|_{L^2\left(I^d, L^2(K_Q)\right)}. \end{aligned}$$

□

If $\mathbf{C}^{(\kappa)} \equiv 0$ for all $\kappa \in \mathcal{K}$, then for any $\hat{b} \in E(K_Q)$ Since $\boldsymbol{\sigma}(t)$ is a positive definite matrix for $t \in I^d$ and $t \neq 0, \boldsymbol{\sigma}^{-1}(t)$ exists for all $t \neq 0$ and

$$\hat{u}(t) = \boldsymbol{\sigma}^{-1}(t)\hat{f}(t).$$

$\boldsymbol{\sigma}^{-1}(t)$ is singular at the origin, and $\hat{u}(t)$ has pole at $t = 0$ in general if $\hat{f}(t) \in L^2(I^d)$(i.e. $f \in L^2(\mathcal{G})$. Therefore, the integral in (2.30) may diverge due to the singularity at the origin. To make the integral converge, we have to impose additional conditions on $\hat{f}(t)$ which are able to absorb the singualrity of $\boldsymbol{\sigma}$ at the origin. For instnace, $\hat{f} \in H^{\nu}_{per}\left(I^d\right)$ with $\nu \geq 1$ and vanishing at $t = 0$,

which is equivalent to $f \in L^2_\nu(\mathcal{G})$ with $\nu \geq 1$ and $\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}} f_{n,\kappa} = 0$. Based on the analysis on the lattice probloems without absolute terms in next section, the following theorem on the existence and uniquness solution and validation of the representation formula can be proved.

Theorem 2.12 If $f \in L^2_\nu(\mathcal{G})$ with $\nu > 1$ for $d = 2$ and $\nu = 1$ for $d = 1,3$, and $\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{K}} f_{n,\kappa} = 0$, then the lattice problem (2.6) with $\mathbf{C}^{(\kappa)} \equiv 0$ has solution, which can be representd by the formula (2.30) and

$$\|u\|_{H^1(\mathcal{G})} = \|\mathcal{F}^{-1}\left(\boldsymbol{\sigma}(t)^{-1}\hat{f}(t)\right)\|_{H^1(\mathcal{G})} \leq C\|\hat{f}\|_{H^\nu_{per}(I^d)} \cong \|f\|_{L^2_\nu(\mathcal{G})}.$$

2.5 Two Examples Of Lattice Problems

We will analyze two concrete lattice problems. One is one-dimensional, and another is two-dimensional. Although the structures of these two problems are simple, the analysis we carry out here can be generalized to other lattice problems.

A lattice problem in one dimension

Suppose elastic rode of two different materials with half-unit length and intersection area A are connected by hinges at nodes, shown in Fig. 2.1. The master cell $Q = [0,1)$, containing two nodes $x^{(\kappa)}, \kappa \in \mathcal{K} = \{1,2\}$. The nodes in cells Q_m for $m \in \mathcal{Z}$ are denoted by $x^{(m,\kappa)}, \kappa = 1,2$. Let $\mathcal{M} = \{x^{(m,\kappa)}, m \in \mathcal{Z}, \kappa = 1,2\}$ denote the global mesh containing all nodes. Suppose that the rods are furnished with springs at each node. By E_1 and E_2 we denote the Young's modulus of the rode, and by $4C_1$ and $4C_2$, the Hook's constant of the springs, respectively. A lattice $\mathcal{G}$ denotes such a structure, connectivity and periodic translation.

Equilibrium Equation

Let u_j and $4f_j$ denotes the displacement of the rods and external force at the nodes x_j, we have the following equilibrium equations

$$\begin{aligned} -A\{E_1(u_{m,2}-u_{m,1}) + E_2(u_{m,1}-u_{m-1,2})\} + C_1u_{m,1} &= f_{m,1} \\ -A\{E_2(u_{m+1,1}-u_{m,2}) + E_1(u_{m,2}-u_{m,1})\} + C_2u_{m,2} &= f_{m,2} \end{aligned} \tag{2.31}$$

Variational Equation

The corresponding variational equation

$$B(u,v) = F(v).$$

where u, v and f are functions defined on $\mathcal{G}$, with the bilinear form

$$\begin{aligned} B(u,v) = & \sum\nolimits_{m\in\mathcal{Z}}(u_{m,2}-u_{m,1})AE_1(v_{m,2}-u_{m,1}) + C_1u_{m,1}v_{m,1} \\ + & \sum\nolimits_{m\in\mathcal{Z}}(u_{m+1,1}-u_{m,2})AE_2(v_{m+1,1}-v_{m,2}) + C_2u_{m,2}v_{m,2} \end{aligned}$$

and the linear functional

$$F(v) = \sum_{m\in\mathcal{Z}}\sum_{j=1,2} f_{m,j}v_{m,j}$$

The energy space $E(\mathcal{G})$ contains functions on $\mathcal{G}$ with finite energy $E(u)$,

$$\begin{aligned} E(u) = \tfrac{1}{2}B(u,u) = & \ \tfrac{1}{2}\textstyle\sum_{m\in\mathcal{Z}} AE_1(u_{m,2}-u_{m,1})^2 + C_1u_{m,1}^2 \\ + & \ \tfrac{1}{2}\textstyle\sum_{m\in\mathcal{Z}} AE_2(u_{m+1,1}-u_{m,2})^2 + C_2u_{m,2}^2. \end{aligned}$$

The spaces $H^1(\mathcal{G})$ and $L^2(\mathcal{G})$ are furnished with the norms

$$\|u\|^2_{H^1(\mathcal{G})} = |u|^2_{H^1(\mathcal{G})} + \|u\|^2_{L^2(\mathcal{G})} = \sum_{m\in\mathcal{Z}} (u_{m,2}-u_{m,1})^2 + (u_{m+1,1}-u_{m,2})^2 + \|u\|^2_{L^2(\mathcal{G})}$$

and

$$\|u\|^2_{L^2(\mathcal{G})} = \sum_{m\in\mathcal{Z}} |u_{m,1}|^2 + |u_{m,2}|^2.$$

Hence, the energy norm $\|u\|_{E(\mathcal{G})} = E(u)^{1/2}$ is equivalent to the norm of the space $H^1(\mathcal{G})$ if $C_1 + C_2 \neq 0$, and equivalent to the semi-norm of the space $H^1(\mathcal{G})$ if $C_1 + C_2 = 0$.

Fourier transform

For $f = f(m), m \in \mathcal{Z}$, we introduce the Fourier transform

$$\mathcal{F}(f) = \hat{f}(t) = \sum_n f(m)e^{imt}, t \in I = (-\pi, \pi)$$

which realizes an isomorphism between $L^2(\mathcal{G})$ and $L^2(I)$, and between $L^2_\nu(\mathcal{G})$ and $H^\nu_(I)$, where the space $L^2_\nu(\mathcal{G})$ is defined as a weighted space with a weighted L^2-norm

$$\|u\|^2_{L^2_\nu(\mathcal{G})} = \sum_{m\in\mathcal{Z}} (1+m^2)^\nu (|u_{m,1}|^2 + |u_{m,2}|^2).$$

The inverse Fourier transform gives $f = \mathcal{F}^{-1}(\hat{f})$ with

$$f(m) = \frac{1}{2\pi}\int_I \hat{f}(t)e^{-imt}dt.$$

Applying the Fourier transform to the equations (2.31), we obtain

$$\begin{aligned} A(E_1+E_2)\hat{u}_1 - A(E_1+E_2e^{it})\hat{u}_2 + C_1\hat{u}_1 &= \hat{f}_1 \\ -A(E_1+E_2e^{-it})\hat{u}_1 + A(E_1+E_2)\hat{u}_2 + C_2\hat{u}_2 &= \hat{f}_2 \end{aligned} \tag{2.32}$$

The corresponding matrix

$$\boldsymbol{\sigma}(t) = \begin{pmatrix} E_{11} & -E_{12} \\ -E_{21} & E_{22} \end{pmatrix}$$

with $E_{11} = A(E_1+E_2)+C_1, E_{22} = A(E_1+E_2)+C_2, E_{12} = A(E_1+E_2e^{it})$ and $E_{21} = A(E_1+E_2e^{-it})$. $\boldsymbol{\sigma}(t)$ is a Hermit matrix, and

$$det(\boldsymbol{\sigma}) = 4A^2E_1E_2sin^2t/2 + A(C_1+C_2)(E_1+E_2) + C_1C_2.$$

Obviously, if $C_1 + C_2 > 0$, $\boldsymbol{\sigma}(t)$ is positive definite for $t \in I$, $\boldsymbol{\sigma}^{-1}(t)$ exists,

$$\boldsymbol{\sigma}^{-1}(t) = \frac{1}{det(\boldsymbol{\sigma})}\begin{pmatrix} E_{02} & E_{21} \\ E_{12} & E_{01} \end{pmatrix}.$$

Representation Formula

If $C_1 + C_2 > 0$, the solution of equation (2.32) can be given by

$$\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} = \boldsymbol{\sigma}^{-1}(t) \begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \end{pmatrix}.$$

Therefore, the solution of the problem (2.31) can be represented as

$$u_\kappa = \mathcal{F}^{-1}(\hat{u}_\kappa(t)) = \mathcal{F}^{-1}(\sum_{1 \le \ell \le 2} \phi_{\kappa,\ell}(t) \hat{f}_\ell(t)),\ \kappa = 1,2$$

with

$$\begin{array}{ll} \phi_{1,1}(t) & = \frac{1}{det(\boldsymbol{\sigma}(t)))} E_{02},\ \phi_{2,2}(t) = \frac{1}{det(\boldsymbol{\sigma}(t)))} E_{01}, \\ \phi_{1,2}(t) & = \frac{1}{det(\boldsymbol{\sigma}(t))} E_{21},\ \phi_{2,1}(t) = \frac{1}{det(\boldsymbol{\sigma}(t))} E_{11}. \end{array}$$

If $C_1 = C_2 = 0, \boldsymbol{\sigma}(t)$ is positive definite for $t \in I$ except $t = 0$, and has a pole of order 2 at $t = 0$. If $|\hat{f}(t)| = O(|t|^\alpha)$ with $\alpha > 1/2$ near the origin, then $\hat{u}$ has a pole of order 1 at $t = 0$. Hence $\hat{u} \notin L^2(I, L^2(K_Q))$, but

$$\langle \boldsymbol{\sigma}(t)\hat{u}, \hat{u} \rangle = \langle \boldsymbol{\sigma}^{-1}(t)\hat{f}, \hat{f} \rangle < \infty$$

which implies that $\|u\|_{E(\mathcal{G})} < \infty$ if $f \in L_1^2(\mathcal{G})$ and $\sum_{m \in \mathcal{Z}} \sum_{\kappa \in \mathcal{K}} f_{m,\kappa} = 0$. Hence the problem (2.31) with $C_1 = C_2 = 0$ has a unique solution in $E(\mathcal{G})$ up to a constant. The argument above based on Fourier transform can be utilized for periodic lattice problems in entire spaces without absolute terms [?, 15], and leads to a result coinciding with Theorem 2.16.

A lattice problem in two dimensions

Suppose elastic rods with intersection area A and unit length are connected by hinges at nodes $x_{k,j} = (k,j) \in \mathcal{Z}^2$, periodically, shown in Fig. 2.2. The master cell $Q = [0,1)^2$, in which there is only one node $x^{(1)}$, and the index set $\mathcal{K} = \{1\}$. The mash $\mathcal{M} = \cup_{m=(k,j) \in \mathcal{Z}^2} x^{(m,1)} = \cup_{(k,j) \in \mathcal{Z}^2} x_{k,j} = \cup_{(k,j) \in \mathcal{Z}^2} (k,j)$. We will use $x_{k,j}$ to denote the nodes in stead of $x^{(m,1)}$. Suppose that the rods are furnished with springs at each node. By E and $\mathbf{C} = diag(C,C)$, we denote the Young's modulus of the rods, and the Hook's constant of the springs, respectively. A lattice $\mathcal{G}$ in two dimensions denotes such a structure, connectivity and periodicity.

Equilibrium Equation

Let $u_{k,j} = (u_{k,j}^{(1)}, u_{k,j}^{(2)})$ and $f_{k,j} = (f_{k,j}^{(1)}, f_{k,j}^{(2)})$ be the displacement vector and external force vector at the node $x_{k,j} = (k,j)$. We have the following Equilibrium equation

$$\begin{array}{ll} AE(-\Delta_1 u_{k,j}^{(1)} + \Delta_1 u_{k-1,j}^{(1)}) & +\frac{1}{2} AE \sum_{\ell=1,2} (-\Delta_{12} u_{k,j}^{(\ell)} + \Delta_{12} u_{k-1,j-1}^{(\ell)}) \\ & +C u_{k,j}^{(1)} = f_{k,j}^{(1)} \\ AE(-\Delta_2 u_{k,j}^{(2)} + \Delta_2 u_{k,j-1}^{(2)}) & +\frac{1}{2} AE \sum_{\ell=1,2} (-\Delta_{12} u_{k,j}^{(\ell)} + \Delta_{12} u_{k-1,j-1}^{(\ell)}) \\ & +C u_{k,j}^{(2)} = f_{k,j}^{(2)} \end{array} \tag{2.33}$$

where

$$\Delta_1 u_{k,j}^{(\ell)} = u_{k+1,j}^{(\ell)} - u_{k,j}^{(\ell)}, \Delta_2 u_{k,j}^{(\ell)} = u_{k,j+1}^{(\ell)} - u_{k,j}^{(\ell)}, \Delta_{12} u_{k,j}^{(\ell)} = (u_{k+1,j+1}^{(\ell)} - u_{k,j}^{(\ell)})/\sqrt{2}.$$

Variational Equation

The corresponding variational equation is

$$B(u,v) = F(v)$$

where the linear functional F and the bilinear form B are defined as

$$F(v) = \sum_{(k,j)\in\mathcal{Z}^2} f_{k,j}^T v_{k,j}$$

and

$$\begin{aligned} B(u,v) \ &= \textstyle\sum_{(k,j)\in\mathcal{Z}^2} (\Delta_1 u_{k,j})^T \mathbf{E} \Delta_1 v_{k,j} + (\Delta_2 u_{k,j})^T \mathbf{E} \Delta_2 v_{k,j} \\ &+ \tfrac{1}{2}(\Delta_{12} u_{k,j})^T \mathbf{E}^* \Delta_{12} u_{k,j} + C u_{k,j}^T v_{k,j} \end{aligned}$$

where $\mathbf{E} = AE\mathcal{I}$ and $\mathbf{E}^* = AE\mathbf{b}\mathbf{b}^T$, $\mathcal{I}$ is an identity 2×2 matrix, and $\mathbf{b}$ is a vector $= (cos\frac{\pi}{4}, sin\frac{\pi}{4})^T$.

Fourier Transform

We introduce the Fourier transform for functions $f = \{f_{k,j}, (k,j) \in \mathcal{Z}^2\}$

$$\mathcal{F}(f) = \hat{f}(t) = \sum_{(k,j)\in\mathcal{Z}^2} f_{k,j} e^{i(kt_1+jt_2)},\ t = (t_1,t_2) \in (-\pi,\pi)^2$$

which leads to an equation in matrix form

$$\boldsymbol{\sigma}(t)\hat{u}(t) = \hat{f}(t) \tag{2.34}$$

where

$$\boldsymbol{\sigma}(t) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

with

$$\begin{aligned} \sigma_{11} \ &= AE(4sin^2\tfrac{t_1}{2} + \sqrt{2}sin^2\tfrac{(t_1+t_2)}{2}) + C \\ \sigma_{12} \ &= \sigma_{21} = -\sqrt{2}AEsin^2\tfrac{(t_1+t_2)}{2} \\ \sigma_{22} \ &= AE(4sin^2\tfrac{t_2}{2} + \sqrt{2}sin^2\tfrac{(t_1+t_2)}{2}) + C \end{aligned}$$

$\boldsymbol{\sigma}(t)$ is a real and symmetric matrix, and

$$\begin{aligned} det(\boldsymbol{\sigma}) = \ & A^2E^2(16sin^2\tfrac{t_1}{2}sin^2\tfrac{t_2}{2} + 4\sqrt{2}sin^2\tfrac{(t_1+t_2)}{2}(sin^2\tfrac{t_1}{2} + sin^2\tfrac{t_2}{2})) \\ & +2AEC(2sin^2\tfrac{t_1}{2} + 2sin^2\tfrac{t_2}{2} + \sqrt{2}sin^2\tfrac{(t_1+t_2)}{2}) + C^2. \end{aligned}$$

If $C > 0$, the matrix $\boldsymbol{\sigma}^{-1}(z)$ is analytic in a strip $\Sigma_\delta = \{z : |Imz| \le \delta\}$ with $\delta > 0$, and $\boldsymbol{\sigma}(t)$ is positive definite for $t \in (-\pi,\pi)^2$, and

$$\boldsymbol{\sigma}^{-1}(t) = \frac{1}{det(\boldsymbol{\sigma})}\begin{pmatrix} \sigma_{22} & -\sigma_{21} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}.$$

Representation Formula

If $C > 0$, then $\boldsymbol{\sigma}(t)$ is positive definite, then we a solution of the equation (2.34)

$$(\hat{u}^{(1)}(t), \hat{u}^{(2)}(t))^T = \boldsymbol{\sigma}^{-1}(t)(\hat{f}^{(1)}(t), \hat{f}^{(2)}(t))^T.$$

Then solution of the equation (2.33) can be represented as

$$\begin{aligned} (u^{(1)}, u^{(2)})^T \ &= \mathcal{F}^{-1}(\boldsymbol{\sigma}^{-1}(t)(\hat{f}^{(1)}(t), \hat{f}^{(2)}(t))^T) \\ &= \mathcal{F}^{-1}(\textstyle\sum_{1\le\ell\le 2} \phi_{1,\ell}\hat{f}^{(\ell)}(t), \sum_{1\le\ell\le 2} \phi_{2,\ell}\hat{f}^{(\ell)}(t))^T. \end{aligned}$$

with

$$\begin{aligned}\phi_{1,1}(t) &= \tfrac{1}{det(\boldsymbol{\sigma})}(AE(4sin^2\tfrac{t_2}{2}+\sqrt{2}sin^2\tfrac{(t_1+t_2)}{2})+C),\\ \phi_{1,2}(t) &= \phi_{2,1}(t)=\tfrac{1}{det(\boldsymbol{\sigma})}\sqrt{2}AEsin^2\tfrac{(t_1+t_2)}{2},\\ \phi_{2,2}(t) &= \tfrac{1}{det(\boldsymbol{\sigma})}(AE(4sin^2\tfrac{t_1}{2}+\sqrt{2}sin^2\tfrac{(t_1+t_2)}{2})+C).\end{aligned}$$

If $C=0, \boldsymbol{\sigma}(t)$ is positive definite for $t\in I^2$ except $t=0$ and has a pole of order 2 at $t=0$. If $|\hat{f}(t)|=O(|t|^\alpha)$ with $\alpha>0$ near the origin, then the equation (2.34) has a solution $\hat{u}(t)\notin L^2(I,L^2(K_Q))$, and has a pole of order 1 at $t=0$. It can be shown that

$$\langle\boldsymbol{\sigma}(t)\hat{u}(t),\hat{u}(t)\rangle=\langle\boldsymbol{\sigma}^{-1}(t)\hat{f}(t),\hat{f}(t)\rangle<\infty$$

which implies that $\|u\|_{E(\mathcal{G})}<\infty$ if $f\in L_1^2(\mathcal{G})$ and $\sum_{m\in\mathcal{Z}^2}\sum_{\kappa\in\mathcal{K}}f_{m,\kappa}=0$ and that the problem (2.33) with $C=0$ has a unique solution in $E(\mathcal{G})$ up to a constant. This coincides with Theorem 2.16

3. UNSTRUCTURED LATTICES IN ENTIRE SPACES

3.1 Non-periodic lattices in entire spaces

A lattices $\mathcal{G}$ in $\mathbf{R}^d, d=1,2,3$ is called unstructured if it is not periodic or quasi-periodic. It is comprised of countable number of nodes distributed in entire spaces, which are connected by elastic rods. Let $K=\{x_k, k\in\mathcal{N}=\{1,2.\dots\}\}$ denote the set of nodes, and let $\mathbf{b}^{(k,\ell)}$ be elastic rod connecting the nodes x_k and x_ℓ with the Young's modules $E^{(k,\ell)}$ and intersect area A. The distribution and connection of the nodes can be periodic or non-periodic. The assumptions on the connectivity and rigidity for periodic lattices in previous section should remain the same, but given in the non-periodic setting. The assumption on connectivity is now decribed as follows:

$(C'.1)$ Each node is connected to others by the rods,at least one node and at most M nodes.

$(C'.2)$ The length of rods are uniformly bounded from below by b_1 and from above by b_2, i.e. for any x_k and x_ℓ which are connected, there holds

$$b_1\le|x_k-x_\ell|\le b_2.$$

$(C'.3)$ Any two nodes x_k and $x_\ell, k,\ell\in\mathcal{N}$ are linked by a chain $L_{k,\ell}$: $x_1^{(k,\ell)}=x_k\to x_2^{(k,\ell)}\to x_3^{(k,\ell)}\to\cdots\to x_s^{(k,\ell)}=x_\ell$ such that $x_t^{(k,\ell)}$ is connected to $x_{t+1}^{(k,\ell)}, 1\le t\le s-1$, and

$$|x_k-x_\ell|\le\sum_{1\le t\le s-1}|x_{t+1}^{(k,\ell)}-x_t^{(k,\ell)}|\le\eta|x_k-x_\ell|.$$

There may be several chains connecting the nodes x_k and x_ℓ, $L_{k,\ell}$ denotes always the shortest one. In particular, $s=1$ if x_k and x_ℓ are connected.

To describe effectively the connection of the nodes, we introduce a set of indices

$$\mathcal{K}_k=\{\ell\in\mathcal{N}|x_\ell \text{ is connected to } x_k\}$$

which together with the assumption (C.1)-(C.3) gives the connectivity of a lattice. The nodes $\{x_k, k \in \mathcal{N}\}$ and the connectivity of the nodes uniquely characterize the structure of a lattice denoted by $\mathcal{G} = \mathcal{G}(K, \mathcal{K}_k)$.

For sake of convenience, we partition the entire space $\mathbf{R}^d$ into cells $Q_m, m = (m_1, m_2, \dots, m_d) \in \mathcal{Z}^d$ such that they are disjointed and $\cup_{m \in \mathcal{Z}^d} Q_m = \mathbf{R}^d$. Here $\mathcal{Z} = \{0, \pm 1, \pm 2 \dots\}$ as in Section 2. Q_m is an interval in $\mathbf{R}^1$, a polygon in $\mathbf{R}^2$, and a polyhedron in $\mathbf{R}^3$. $\bar{Q}_m \cap \bar{Q}_n$ is a vertex, or an edge, or a face, or empty set for any $n, m \in \mathcal{N}, n \neq m$, where $\bar{Q}_m$ denotes the closure of Q_m. Furthermore, a partition of $\mathbf{R}^d$ is assumed to satisfy

(Q.1) Vertices of Q_m are nodes of $\mathcal{G}$, and edges of Q_m are elastic rods;

(Q.2) Every node must be a vertex of a cell, or is located on an edge of a cell, or in interior of a cell, and each node is shared by at most n_Q cells;

(Q.3) The diameter of cells is uniformly bounded,

$$\hat{d}_Q \leq diag(Q_m) \leq d_Q;$$

(Q.4) The number q_m of nodes in each Q_m is uniformly bounded by q;

(Q.5) Each node in a cell is connected to at least one node in the same cell;

(Q.6) Each cell is rigid. According to [1], there is a sufficient condition for the rigidity nof a cell,

$$e_m \geq dq_m - d(d+1)/2$$

where e_m be the number of rods in the cell $\bar{Q}_m$, which leads to rigidity of lattices.

We may introduce a local number $(m, \kappa), \kappa \in \mathcal{N}_m = \{1, 2, \dots, q_m\}$ for the node x_k in the cell Q_m such that $x_k = x^{(m,\kappa)}$ and $|x_k|^2 \cong |m|^2 = \sum_{1 \leq i \leq d} |m_i|^2$. The connection of nodes can described locally described by

$$B_\kappa^m = \{(n, \lambda) \mid x^{(n+m,\lambda)} \text{ is connected to } x^{(m,\kappa)}\}.$$

Proposition 3.1. If $x_k = x^{(m,\kappa)}$ and $x_\ell = x^{(m,\lambda)}$ are in the same cell Q_m, the number of nodes on the chain $L_{k,\ell}$ is not more than $\dfrac{\eta d_Q}{b_1}$.

Proof. Due to $(C'.3)$, there is a chain $L_{k,\ell}$ linking the nodes x_k and $x_\ell : x_{k,\ell}^1 = x_k \to x_{k,\ell}^2 \to x_{k,\ell}^3 \to \cdots \to x_{k,\ell}^s = x_\ell$. By (C.2),(C.3),and (Q.3), there holds

$$sb_1 \leq \sum_{1 \leq t \leq s-1} |x_{t+1}^{(k,\ell)} - x_t^{(k,\ell)}| \leq \eta |x_k - x_\ell| \leq \eta d_Q,$$

which leads to the assertion of the proposition. □

Let $u_k = (u_k^1, u_k^2, \dots, u_k^s)^\top$ be a vector function on the nodes $\{x_k, k \in \mathcal{N}\}$. As in previous section, we shall deal with the problems with $s = 1$ for $1 \leq d \leq 3$ and the truss problems with non-rigid joint in one, two and three dimensions for which $s = d = 1, 2, 3$. We furnish the rods with springs in the axis directions at each node with Hook's coefficients denoted by diagonal matrices $\mathbf{C}^k, k \in \mathcal{N}$.

We assume that the ratio of the length of the rods and the intersect area A of rods >> 1.

If external forces exert on the rods at the nodes x_k, denoted by $f = \{f_k, k \in \mathcal{N}\}$, we have an equilibrium equation for the elastic rod problem

$$-\sum_{\ell \in \mathcal{K}_k} \mathbf{E}^{(k,\ell)} \frac{(u_\ell - u_k)}{|x_\ell - x_k|^2} + \mathbf{C}^k u_k = f_k, \forall k \in \mathcal{N} \tag{3.1}$$

where

$$\mathbf{E}^{(k,\ell)} = AE^{(k,\ell)} \frac{(x_\ell - x_k)}{|x_\ell - x_k|^2} \frac{(x_\ell - x_k)^\top}{|x_\ell - x_k|^2}$$

and

$$\mathbf{C}^k = diag(C_1^k, C_2^k, \ldots, C_d^k), \quad C_l^k \geq 0, 1 \leq l \leq d.$$

Let $H^1(\mathcal{G})$ and $L^2(\mathcal{G})$ be the discrete Sobolev spaces with the norms

$$\|u\|_{L^2(\mathcal{G})}^2 = \sum_{k \in \mathcal{N}} |u_k|^2 \tag{3.2}$$

and

$$\|u\|_{H^1(\mathcal{G})}^2 = |u|_{H^1(\mathcal{G})}^2 + \|u\|_{L^2(\mathcal{G})}^2 \tag{3.3a}$$

where $|u|_{H^1(\mathcal{G})}$ is the semi-norm,

$$|u|_{H^1(\mathcal{G})}^2 = \sum_{k \in \mathcal{N}} \sum_{\ell \in \mathcal{K}_k} \frac{|u_\ell - u_k|^2}{|x_\ell - x_k|^2}. \tag{3.3b}$$

Similarly, the corresponding variational equation can be derived

$$B(u, v) = F(v) \forall v \in H^1(\mathcal{G}) \tag{3.4}$$

where

$$B(u, v) = \sum_{k \in \mathcal{N}} \left\{ \sum_{\ell \in \mathcal{K}_k} \frac{1}{2} \langle \mathbf{E}^{(k,\ell)} \frac{u_\ell - u_k}{|x_\ell - x_k|}, \frac{v_\ell - v_k}{|x_\ell - x_k|} \rangle + \langle \mathbf{C}^k u_k, v_k \rangle \right\} \tag{3.5a}$$

is a bilinear form on $H^1(\mathcal{G}) \times H^1(\mathcal{G})$, and

$$F(v) = \sum_{k \in \mathcal{N}} \langle f_k, v_k \rangle. \tag{3.5b}$$

is a linear functional on $H^1(\mathcal{G})$.

The energy of the lattice $\mathcal{G}$ is defined as

$$G(u) = B(u, u) = \sum_{k \in \mathcal{N}} \{ \sum_{\ell \in \mathcal{K}_k} \frac{1}{2} \langle \mathbf{E}^{(k,\ell)} \frac{u_\ell - u_k}{|x_\ell - x_k|}, \frac{u_\ell - u_k}{|x_\ell - x_k|} \rangle + \langle \mathbf{C}^k u_k, u_k \rangle \} \tag{3.6}$$

The energy space denoted by $E(\mathcal{G})$ is the family of all grid functions u on $\mathcal{G}$ with finite energy $G(u)$ defined in (3.6), and $G(u)^{1/2}$ is referred as the energy norm $\|u\|_{E(\mathcal{G})}$. The assumption (Q.6) leads to the rigidity of lattices, and

$$\sum_{k \in \mathcal{N}} \sum_{\ell \in \mathcal{K}_k} \langle \mathbf{E}^{(k,\ell)} \frac{u_\ell - u_k}{|x_\ell - x_k|}, \frac{u_\ell - u_k}{|x_\ell - x_k|} \rangle = 0. \tag{3.7}$$

if and only if u is a rigid body motion.

At the node $x_k = x^{(m,\kappa)}$, we may write $u_k = u_{m,\kappa}, \mathbf{C}^k = \mathbf{C}^{(m,\kappa)}, f_k = f_{m,\kappa}$, etc. Due to the disjointedness of cells, we have the following proposition which allows us to write the norms either in global numbering or in local numbering.

Proposition 3.2 For $u \in L^2(\mathcal{G})$, $v \in H^1(\mathcal{G})$ and $w \in E(\mathcal{G})$, there hold, respectively,

$$\|u\|^2_{L^2(\mathcal{G})} = \sum_{m\in\mathcal{N}} \sum_{\kappa\in\mathcal{N}_m} |u_{m,\kappa}|^2,$$

$$\|v\|^2_{H^1(\mathcal{G})} = \sum_{m\in\mathcal{N}} \sum_{\kappa\in\mathcal{N}_m} \sum_{(n,\lambda)\in\mathcal{K}^m_\kappa} \frac{|v_{n,\lambda} - v_{m,\kappa}|^2}{|x^{(n,\lambda)} - x^{(m,\kappa)}|^2},$$

and

$$\begin{aligned}\|w\|^2_{E(\mathcal{G})} &= \tfrac{1}{2} \textstyle\sum_{m\in\mathcal{N}} \sum_{\kappa\in\mathcal{N}_m} \sum_{(n,\lambda)\in\mathcal{K}^m_\kappa} \langle \mathbf{E}^{(k,\ell)} \dfrac{w_{n,\lambda} - w_{m,\kappa}}{|x^{(n,\lambda)} - x^{(m,\kappa)}|}, \dfrac{w_{n,\lambda} - w_{m,\kappa}}{|x^{(n,\lambda)} - x^{(m,\kappa)}|}\rangle \\ &\quad + \textstyle\sum_{m\in\mathcal{N}} \sum_{\kappa\in\mathcal{N}_m} \langle \mathbf{C}^{(m,\kappa)} w_{m,\kappa}, w_{m,\kappa}\rangle.\end{aligned}$$

As analogue of the Theorem 2.3, we have the existenc and uniqueness for the non-structured lattice problems (3.1).

Theorem 3.3 The variational form B on $H^1(\mathcal{G}) \times H^1(\mathcal{G})$ given in (3.6) is continuous and coercive if $\mathbf{C}^k \not\equiv 0$ in the sense that there exists a node $x_{k_m} = x^{(m,\kappa_m)}$ in each cell such that $\mathbf{C}^{k_m} \neq 0$, and the problem (3.1) has a unique solution $u \in H^1(\mathcal{G})$.

Proof The proof is analogous to that for periodic lattice problem (2.6). For the detail in the non-periodic setting, we refer to [?].

3.3 Problems Without Absolute Terms

The proof for the existence and uniqueness of solutions for non-periodic lattices with absolute terms is similar to those for periodic lattices which we have invetigated intensively. We should concentrate on the problem of non-periodic lattices without absolute terms in this section for which the Fourier transform can not be used. Fourier transform is an extremely powerful tool for periodic lattices with and without absolute terms. It is worth to pointing out that in practical applications, problems of structured or unstructured lattices are associated with no absolute term, i.e. $\mathbf{C}^{(k)} \equiv 0$ for all $k \in \mathcal{N}$. We need to develop a new approach for analysis of non-periodic lattices without absolute terms,which is extremely important practically and theoretically

We seek $u \in E(\mathcal{G})$ such that

$$B(u,v) = F(v)\ \forall v \in E(\mathcal{G}) \tag{3.8}$$

where

$$B(u,v) = \sum_{k\in\mathcal{N}} \sum_{\ell\in\mathcal{K}_k} \frac{1}{2} \langle \mathbf{E}^{(\kappa,\ell)} \frac{u_\ell - u_k}{|x_\ell - x_k|}, \frac{v_\ell - v_k}{|x_\ell - x_k|}\rangle \tag{3.9a}$$

and

$$F(v) = \langle f, v\rangle_{\mathcal{G}} = \sum_{k\in\mathcal{N}} \langle f_k, v_k\rangle. \tag{3.9b}$$

For the problem (3.8), the energy space $E(\mathcal{G})$ does not coincide the $H^1(\mathcal{G})$, and $E(\mathcal{G})$ is not embedded in $L^2(\mathcal{G})$. Actually, the "energy norm" $\|u\|_{E(\mathcal{G})}$ is equivalent to the semi-norm $|u|_{H^1(\mathcal{G})}$. Consequently, the solution of the lattice problem is not unique and may not exist in $H^1(\mathcal{G})$. Hence we have to modify the space $H^1(\mathcal{G})$, such that the solution exists uniquely in modified space $\tilde{H}^1(\mathcal{G})$ with the norm equivalent to semi-norm of $H^1(\mathcal{G})$.

The modification of the spaces and the equivalence of the norms for one dimension is different from those for two and three dimensions. We shall address them separately. To this end, we define a weighted space $L^2_{\nu,\sigma}(\mathcal{G})$ for all dimensions with the norm

$$\begin{aligned} \|u\|^2_{L^2_{\nu,\sigma}(\mathcal{G})} &= \textstyle\sum_{k\in\mathcal{N}} (1+|x_k|^2)^\nu \log^{2\sigma}(1+|x_k|)\,|u_k|^2 \\ &\cong \textstyle\sum_{m\in\mathcal{Z}^d}\sum_{\kappa\in\mathcal{N}_m} (1+|m|^2)^\nu \log^{2\sigma}(1+|m|)\,|u_{m,\kappa}|^2 \end{aligned} \tag{3.10}$$

where ν and σ are real numbers. We shall write $L^2_{0,0}(\mathcal{G}) = L^2(\mathcal{G}), L^2_{\nu,0}(\mathcal{G}) = L^2_\nu(\mathcal{G})$. Obviously, $L^2_{\nu,\sigma}(\mathcal{G}) \supseteq L^2(\mathcal{G})$ if $\nu,\sigma \le 0$, and $L^2_{\nu,\sigma}(\mathcal{G}) \subseteq L^2(\mathcal{G})$ if $\nu,\sigma \ge 0$.

3.3.1 Problems without absolute terms in one dimension

Lemma 3.4 If $v \in E(\mathcal{G})$, and $v_0 = 0$, then

$$\|v\|_{L^2_{-1}(\mathcal{G})} \le C|v|_{H^1(\mathcal{G})}. \tag{3.11}$$

Proof. Due to the proposition 3.2, we shall use local numbering for the nodes. First, suppose that $x^{(m,\kappa)}$ and $x^{(m,1)}$ in the cell Q_m are always connected for any $\kappa \in \mathcal{N}_m$ with $\kappa \ne 1$. Then

$$\begin{aligned} \textstyle\sum_{m\in\mathcal{Z}}\sum_{\kappa\in\mathcal{N}_m} \frac{|v_{m,\kappa}|^2}{1+m^2} &\le C\left(\textstyle\sum_{m\in\mathcal{Z}}\sum_{\kappa\in\mathcal{K}} (v_{m,\kappa}-v_{m,1})^2 + \sum_{m\in\mathcal{Z}} \frac{|v_{m,1}|^2}{1+m^2}\right) \\ &\le C\left(|v|^2_{H^1(\mathcal{G})} + \textstyle\sum_{m\in\mathcal{Z}} \frac{|v_{m,1}|^2}{1+m^2}\right). \end{aligned} \tag{3.12}$$

If there are some $\kappa \in \mathcal{N}_m$ such that $x^{(m,\kappa)}$ and $x^{(m,1)}$ are not connected, there always exists by $(C'.3)$ a chain $L_{m,1,m,\kappa} : x^{(m,1)} = x^{(n_1,\lambda_1)} \to x^{(n_2,\lambda_2)} \ldots \to x^{(n_s,\lambda_s)} = x^{(m,\kappa)}$, and due to Proposition 3.1

$$\sum_{m\in\mathcal{Z}} |v_{m,\kappa}-v_{m,1}|^2 \le C\sum_{m\in\mathcal{Z}}\sum_{\kappa\in\mathcal{N}_m}\sum_{(n,\lambda)\in B^m_{\mathcal{K}}} |x^{(m+n,\lambda)} - x^{(m,\kappa)}|^2 \le C|v|^2_{H^1(\mathcal{G})}. \tag{3.13}$$

Hence, (3.12) holds for the cases that $x^{(m,\kappa)}$ and $x^{(m,1)}$ are connected or not connected.

Let $w_m = v_{m,1}$. It is sufficient to show that

$$\sum_{m=1}^{\infty} \frac{|w_m|^2}{1+m^2} \le C\sum_{m=1}^{\infty} |w_m - w_{m-1}|^2 \tag{3.14}$$

We assume that $v_0 = v_{0,1} = w_0 = 0$. We have by Cauchy inequality

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{|w_m|^2}{1+m^2} &\leq \sum_{m=1}^{\infty} \frac{1}{1+m^2} \left(\sum_{j=1}^{m} (w_j - w_{j-1}) \right)^2 \\ &\leq \sum_{m=1}^{\infty} \frac{1}{1+m^2} \left(\sum_{j=1}^{m} (w_j - w_{j-1})^2 j^{\varepsilon} \right) \left(\sum_{j=1}^{m} j^{-\varepsilon} \right)\end{aligned}$$

where $\varepsilon \in (0,1)$, arbitrary. Note that

$$\sum_{j=1}^{m} j^{-\varepsilon} \leq C \int_1^m \xi^{-\varepsilon} d\xi \leq C m^{1-\varepsilon}$$

which implies that

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{|w_m|^2}{1+m^2} &\leq C \sum_{m=1}^{\infty} \frac{m^{1-\varepsilon}}{1+m^2} \sum_{j=1}^{m} (w_j - w_{j-1})^2 j^{\varepsilon} \\ &\leq C \sum_{j=1}^{\infty} (w_j - w_{j-1})^2 j^{\varepsilon} \sum_{m=j}^{\infty} \frac{m^{-\varepsilon}}{1+m^2}.\end{aligned}$$

For $j \leq 1$, there holds

$$\sum_{m=j}^{\infty} \frac{m^{-\varepsilon}}{1+m^2} \leq C \sum_{m=j}^{\infty} m^{-1-\varepsilon} \leq C \int_j^{\infty} \xi^{-1-\varepsilon} d\xi \leq C j^{-\varepsilon}$$

which leads to (3.14). Therefore, there holds

$$\sum_{m=1}^{\infty} \frac{|v_{m,1}|^2}{1+m^2} \leq C \sum_{m=1}^{\infty} |v_{m,1} - v_{m-1,1}|^2 .$$

Similarly, it is true for $m \leq 0$. Therefore we have

$$\sum_{m \in \mathcal{Z}} \frac{|v_{m,1}|^2}{1+m^2} \leq C \sum_{m \in \mathcal{Z}} |v_{m,1} - v_{m-1,1}|^2 . \tag{3.15}$$

Note that $v_{m,1}$ and $v_{m-1,1}$ may be connected or not connected, the argument for (3.13) can be carried once more here. Hence, we have

$$\sum_{m \in \mathcal{Z}} |v_{m,1} - v_{m-1,1}|^2 \leq C \sum_{m \in \mathcal{Z}} \sum_{\kappa \in \mathcal{N}_m} \sum_{(n,\lambda) \in B_\kappa^m} |v_{m+n,\lambda} - v_{m,\kappa}|^2 \leq C |v|^2_{H^1(\mathcal{G})} ,$$

which together with (3.12) and (3.15) leads to (3.11). □

Theorem 3.5 If $f \in L_1^2(\mathcal{G})$ and $\sum_{k \in \mathcal{N}} f_k = 0$, then for any $v \in E(\mathcal{G})$ it holds that

$$|\langle f, v \rangle_{\mathcal{G}}| = \left| \sum_{k \in \mathcal{N}} f_k v_k \right| \leq c \|f\|_{L_1^2(\mathcal{G})} |v|_{H^1(\mathcal{G})} \tag{3.16}$$

Proof. Note that $L^1(\mathcal{G}) \subset L^2_1(\mathcal{G})$ and the sum $\sum_{k\in\mathcal{N}} f_k$ exists. By Lemma 3.4 we have

$$\begin{aligned}|\langle f, v\rangle_{\mathcal{G}}| &= \left|\sum_{k\in\mathcal{N}} f_k v_k\right| = \left|\sum_{k\in\mathcal{N}} f_k\,(v_k - v_0)\right| \\ &\leq \left(\sum_{k\in\mathcal{N}} \left(1+|x_k|^2\right)|f_k|^2\right)^{1/2} \left(\sum_{k\in\mathcal{N}} \frac{(v_k-v_0)^2}{1+|x_k|^2}\right)^{1/2} \\ &\leq C\|f\|_{L^2_1(\mathcal{G})}|v|_{H^1(\mathcal{G})}.\end{aligned}$$

□

We introduce the space $\tilde{H}^1(\mathcal{G})$ with the norm

$$\|u\|_{\tilde{H}^1(\mathcal{G})} = \left\{\|u\|^2_{L^2_{-1}(\mathcal{G})} + |u|^2_{H^1(\mathcal{G})}\right\}^{1/2},$$

and a quotient space

$$\hat{H}^1(\mathcal{G}) = \tilde{H}^1(\mathcal{G})/\mathcal{P}_r$$

with norm

$$\|u\|_{\hat{H}^1(\mathcal{G})} = \inf_{\alpha\in P_c}\|u-\alpha\|_{\tilde{H}^1(\mathcal{G})}$$

where $\mathcal{P}_r$ is the set of all constant functions on $\mathcal{G}$, which is a subspace of $\tilde{H}^1(\mathcal{G})$. Then, by Lemma 3.4 ,

$$\|u\|_{\hat{H}^1(\mathcal{G})} \cong |u|_{H^1(\mathcal{G})} \cong \|u\|_{E(\mathcal{G})}.$$

In the framework of the space $\tilde{H}^1(\mathcal{G})$ and $\hat{H}^1(\mathcal{G})$ we are addressing the existence and uniqueness of the solution of the problem (3.8) with $d = 1$.

Theorem 3.6 If $f \in L^2_1(\mathcal{G})$ and $\sum_{k\in\mathcal{N}} f_k = 0$, then the problem (3.8) with $d = 1$ has a solution $u \in E(\mathcal{G})$, and

$$|u|_{H^1(\mathcal{G})} \leq C\|f\|_{L^2_1(\mathcal{G})}. \tag{3.17}$$

The solution is unique up to a constant.

Proof. Due to the equivalence between $|u|_{H^1(\mathcal{G})}$ and $\|u\|_{\hat{H}^1(\mathcal{G})}$

$$\begin{aligned}|B(u,v)| &\leq C|u|_{H^1(\mathcal{G})}|v|_{H^1(\mathcal{G})} \\ &\leq C\|u\|_{\hat{H}^1(\mathcal{G})}\|v\|_{\hat{H}(\mathcal{G})}\end{aligned}$$

and

$$|B(u,u)| \geq D|\hat{u}|^2_{H^1(\mathcal{G})} \geq D\|u\|_{\hat{H}^1(\mathcal{G})}.$$

By Theorem 3.5, it holds that

$$|F(v)| \leq C\|f\|_{L^2_1(\mathcal{G})}\|v\|_{\hat{H}^1(\mathcal{G})}.$$

By Lax-Milgram Theorem, the variational problem has a unique solution $u \in \hat{H}^1(\mathcal{G})$, and

$$\|u\|_{\hat{H}^1(\mathcal{G})} \leq C\|f\|_{L^2_1(\mathcal{G})},$$

which implies (3.17) and the uniqueness of the solution in $E(\mathcal{G})$ up to a constant.

3.3.2 Problems without absolute terms in two dimensions

With suitable spaces $L^2_{\nu,\sigma}(\mathcal{G})$ and $\tilde{H}^1(\mathcal{G})$ we are able to address properly the problem (3.8) without absolute term in two dimensions. The next theorem is essential to the existence and uniqueness of solutions.

Theorem 3.7 If $f \in L^2_{1,1}(\mathcal{G})$, and $\sum_{k\in\mathcal{N}} f_k = 0$, then for any $v \in E(\mathcal{G})$,

$$|\langle f, v\rangle_{\mathcal{G}}| \le C\|f\|_{L^2_{1,1}(\mathcal{G})}|v|_{H^1(\mathcal{G})} \tag{3.18}$$

with constant C independent of f and v.

The theorem is parallel to Theorem 3.6 for one dimension, but the proof for two dimensions needs some embedding results for functions in $\mathbf{R}^2$, which is contained in Appendix A. In order to use these results, we have to extend by linear interpolation grid functions on $\mathcal{G}$ to whole space $\mathbf{R}^2$.

Let $\bar{K}_m$ be a set of nodes which are located in $\bar{Q}_m$ where $\bar{Q}_m$ is the closure of the cell Q_m. Obviously, $x^{(m,\kappa)} \in \bar{K}_m$ for all $\kappa \in \mathcal{K}$, and some nodes $x^{(n,\kappa)}$ in neighboring cells are included as well. Let K^V_m, K^I_m and K^E_m be subsets of $\bar{K}_m$ for nodes at vertices, on edges (not including vertices) and in the interior of Q_m, respectively. Then $\bar{K}_m = K^E_m \cup K^V_m \cup K^I_m$.

By $\mathcal{T}_m = \{t_i, i = 1, 2, \ldots, T\}$, we denote a triangular partition of $\bar{Q}_m$ satisfying the following conditions:

(T.1)$V_{\mathcal{T}_m} = \bar{K}_m$, where $V_{\mathcal{T}_m}$ denote a set of all vertices of the partition $\mathcal{T}_m$;

(T.2) The partition is regular, i.e. $t_i \cap t_j$ for $i \neq j$ is a vertex, or a whole edge, or empty.

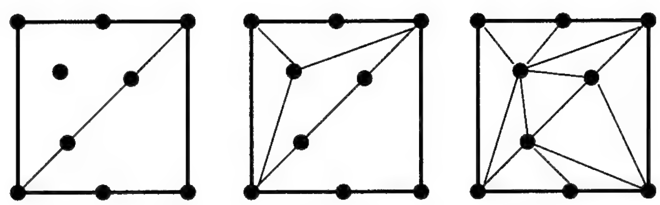


Fig. 3.1 Triangular partition of cell $\bar{Q}_m$

The construction of such a partition can be started with an initial partition $\mathcal{T}^0_m$ of $\bar{Q}_m$ for which (T.2) holds and $V_{\mathcal{T}^0_m} = K^V_m$, shown in Fig. 3.1(a). For the partition $\mathcal{T}^0_m$ there may be some nodes in the interior of triangles t_is. If a node $x^{(n,\kappa)} \in K^I_m$ is in the interior of t_i, we divide t_i into three smaller triangles by connecting $x^{(n,\kappa)}$ to four vertices of t_i, shown in Fig. 3.1(b). Repeating the process for each node in the interiors of all simplices, we have a partition $\mathcal{T}^1_m$ of $\bar{Q}_m$ for which (T.2) holds and no node $x^{(n,\kappa)} \in \bar{K}_m$ is located in the interiors of all triangles. Note that nodes $x^{(n,\kappa)} \in K^I_m \cup K^E_m$ may be located on an edge of t_i in the partition $\mathcal{T}^1_m$. Suppose there are l nodes are on an (open)edge of a triangle t_i, we divide the triangle t_i into $l + 1$ smaller triangles by connecting these l nodes and the vertex opposite to the edge, shown in Fig. 3.1(c). If

this edge is shared by a pair of triangles t_i and t_j, we divide each of these two triangles into $l+1$ smaller triangles. Applying this process to each edge of the triangles will results in a triangular partition $\mathcal{T}_m$ satisfying (T.1) and (T.2).

The triangular partition $\mathcal{T}_m$ of the cell $\bar{Q}_m$ can be individually carried out, a combination of the triangular partitions $\mathcal{T}_m$ for all $m \in Z^2$ forms a partition $\mathcal{T}$ of $\mathbf{R}^2$, and $V_{\mathcal{T}} = \cup_{m\in Z^2}\bar{K}_m = \cup_{k\in\mathcal{N}} x_k$ where $V_{\mathcal{T}}$ is the set of all vertices of $\mathcal{T}$.

Based on such a partition $\mathcal{T}$, we can extend a grid function u on $\mathcal{G}$ to a function $\tilde{u}(x)$ for $x \in \mathbf{R}^2$ by a linear interpolation. Let $\phi_i(x)$ be a linear function in t_i such that $\phi_i(x^{(n,\kappa)}) = u_{n,\kappa}$ at all vertices of t_i, and let $\psi_m(x)$ be a piecewise linear function in $\bar{Q}_m$ such that $\psi_m(x) = \phi_i(x)$ in $t_i, 1 \le i \le T$. Then, there holds

$$
\begin{aligned}
|\psi_m(x)|^2_{H^1(Q_m)} &= \textstyle\sum_{1\le i\le T} |\phi_i(x)|^2_{H^1(t_i)} \\
&\le C \textstyle\sum_{t_i\subseteq\bar{Q}_m} \sum_{x^{(n,\kappa)},x^{(l,\lambda)}\in t_i} \dfrac{|u_{n,\kappa}-u_{l,\lambda}|^2}{|x^{(n,\kappa)}-x^{(l,\lambda)}|^2} \\
&\le C \textstyle\sum_{t_i\subseteq\bar{Q}_m} \sum_{x^{(n,\kappa)},x^{(l,\lambda)}\in t_i} |u_{n,\kappa}-u_{l,\lambda}|^2.
\end{aligned}
$$

Let $\tilde{u}(x) = \psi_m(x)$ in $\bar{Q}_m$ for all $m \in \mathcal{Z}^2$. Then, $\tilde{u}(x)$ is continuous and piecewise linear function in $\mathbf{R}^2$, and

$$
|\tilde{u}|^2_{H^1(R^2)} \le C \sum_{m\in\mathcal{Z}^2} \sum_{t_i\subseteq\bar{Q}_m} \sum_{x^{(n,\kappa)},x^{(l,\lambda)}\in t_i} |u_{n,\kappa}-u_{l,\lambda}|^2. \tag{3.19}
$$

Note that if the vertices $x^{(n,\kappa)}$ and $x^{(l,\lambda)}$ of t_i in $\bar{Q}_m$ are not connected, due to $(C'.2)$ and Proposition 3.1, they are linked by the shortest chain $L_{n,\kappa,l,\lambda}$: $x^{(n_1,\lambda_1)} = x^{(n,\kappa)} \to x^{(n_2,\lambda_2)} \cdots \to x^{(n_s,\lambda_s)} = x^{(l,\lambda)}$ with s uniformly bounded, where the node $x^{(n_j,\lambda_j)}$ is connected to the node $x^{(n_{j+1},\lambda_{j+1})}$ for $1 \le j \le s-1$. Hence, there holds

$$
|u_{n,\kappa} - u_{l,\lambda}| \le C \sum_{1\le j\le s-1} |u_{n_j,\lambda_j} - u_{n_{j+1},\lambda_{j+1}}|
$$

which with (3.19) implies that

$$
|\tilde{u}|^2_{H^1(R^2)} \le C \sum_{m\in\mathcal{Z}^2} \sum_{\kappa\in\mathcal{N}_m} \sum_{(n,\lambda)\in B^m_\kappa} |u_{m,\kappa} - u_{n+m,\lambda}|^2. \tag{3.20}
$$

Theorem 3.8 Let u be a grid function on the lattice $\mathcal{G}$, and let $\tilde{u}$ be the extension of u by a linear interpolation, described as above. Then $|u|_{H^1(\mathcal{G})} \cong |\tilde{u}|_{H^1(\mathbf{R}^2)}$, and $\|u\|_{L^2_{\nu,\sigma}(\mathcal{G})} \cong \|\tilde{u}\|_{L^2_{\nu,\sigma}(\mathbf{R}^2)}$, i.e. there are two positive constants C_1 and C_2 independent of u and $\tilde{u}$ such that

$$
C_1|u|_{H^1(\mathcal{G})} \le |\tilde{u}|_{H^1(\mathbf{R}^2)} < C_2|u|_{H^1(\mathcal{G})} \tag{3.21}
$$

and

$$
C_1\|u\|_{L^2_{\nu,\sigma}(\mathcal{G})} \le \|\tilde{u}\|_{L^2_{\nu,\sigma}(\mathbf{R}^2)} < C_2\|u\|_{L^2_{\nu,\sigma}(\mathcal{G})} \tag{3.22}
$$

Proof. Since $\tilde{u}(x)$ is a piecewise linear function interpolating the grid function u at each node $x^{(m,\kappa)}$, and for $x \in \bar{Q}_m$,

$$(1+x^2)^\nu \log^{2\sigma}(1+|x|) \cong (1+|m|^2)^\nu \log^{2\sigma}(1+|m|),$$

it holds that

$$\|\tilde{u}(x)\|^2_{L^2_{\nu,\sigma}(\bar{Q}_m)} \cong \sum_{x^{(n,\kappa)} \in \bar{K}_m} (1+|m|^2)^\nu \log^{2\sigma}(1+|m|)|u_{n,\kappa}|^2$$

which implies (3.22).

The second inequality of (3.21) follows from (3.20). It suffices to show the first inequality of (3.16). Suppose that $x^{(m,\kappa)}$ and $x^{(n+m,\lambda)}$ are connected. If $x^{(m,\kappa)}$ and $x^{(n+m,\lambda)}$ are in the cell $\bar{Q}_m$, it is easy to see that

$$|u_{m,\kappa} - u_{n+m,\lambda}| \leq C|\psi_m|_{H^1(Q_m)}. \tag{3.23}$$

We next consider two connected nodes $x^{(m,\kappa)}$ and $x^{(n+m,\lambda)}$ which are located in different cells $\bar{Q}_m$ and $\bar{Q}_n$. Let $Q_{n_j}, 1 \leq j \leq J$ be a sequence of cells with $Q_{n_1} = Q_m$ and $Q_{n_J} = Q_n$ such that Q_{n_j} is neighboring to $Q_{n_{j+1}}$. Due to the assumption $(C'.3)$, J is uniformly bounded. Select a common vertex $x^{(n_j,\lambda_j)}$ of the cell $\bar{Q}_{n_j}$ and $\bar{Q}_{n_{j+1}}, 1 \leq j \leq J-1$. Therefore, we have

$$\begin{aligned} |u_{m,\kappa} - u_{n+m,\lambda}| \;&\leq |u_{m,\kappa} - u_{n_1,\lambda_1}| + |u_{m+n,\lambda} - u_{n_J,\lambda_J}| \\ &\quad + \textstyle\sum_{1\leq j\leq J-1} |u_{n_j,\lambda_j} - u_{n_{j+1},\lambda_{j+1}}| \end{aligned} \tag{3.24}$$

Since $x^{(n_j,\lambda_j)}$ and $x^{(n_{j+1},\lambda_{j+1})}$ are in the same cell $\bar{Q}_{n_{j+1}}$, we have for $1 \leq j \leq J-2$

$$|u_{n_j,\lambda_j} - u_{n_{j+1},\lambda_{j+1}}| \leq C|\psi_{n_{j+1}}|_{H^1(Q_{n_{j+1}})}, \tag{3.25a}$$

Similarly, there hold

$$|u_{m,\kappa} - u_{n_1,\lambda_1}| \leq C|\psi_m|_{H^1(Q_m)}, \tag{3.25b}$$

and

$$|u_{m+n,\lambda} - u_{n_{J-1},\lambda_{J-1}}| \leq C|\psi_n|_{H^1(Q_n)}. \tag{3.25c}$$

A combination of (3.24) and (3.25) leads to

$$|u_{m,\kappa} - u_{n+m,\lambda}| \leq C \sum_{1\leq j\leq J} |\psi_{n_j}|_{H^1(Q_{n_j})}. \tag{3.26}$$

The first inequality of (3.21) follows easily from (3.23) and (3.26). $\square$

Lemma 3.9 For $u \in \tilde{H}^1(\mathcal{G})$ there exists a constant α such that

$$\|u - \alpha\|_{L^2_{-1,-1}(\mathcal{G})} \leq C|u|_{H^1(\mathcal{G})} \tag{3.27}$$

Proof Let $\tilde{u}(x)$ be the extension described above and $\alpha = \int_\Gamma \tilde{u}(x)dx$ where $\Gamma = \{x \in \mathbf{R}^2 \mid |x| = 2\}$. By Theorem A.1 there holds

$$\int_S |\tilde{u} - \alpha|^2 \, dx + \int_{S^c} \frac{|\tilde{u} - \alpha|^2}{|x|^2 \log^2 |x|} dx \leq C|\tilde{u}|^2_{H^1(\mathbf{R}^2)}$$

where $S = \{x \in \mathbf{R}^2 \mid |x| \leq 2\}, S^c = \mathbf{R}^2 \setminus S$. This estimation and Theorem 3.8 lead to (3.27). $\square$

We are now able to prove Theorem 3.7.

Proof of Theorem 3.7 Let $\tilde{v}(x)$ be the extension of v, and $\alpha = \int_\Gamma \tilde{v}(x)dx$ where $\Gamma = \{x \in \mathbf{R}^2 \mid |x| = 2\}$. Since $\sum_{k\in\mathcal{N}} f_k = 0$, we have by Theorem 3.8 and Lemma 3.9

$$\begin{aligned} |\langle f, v\rangle_{\mathcal{G}}| &= |\langle f, v-\alpha\rangle_{\mathcal{G}}| \\ &\leq C\|f\|_{L^2_{1,1}(\mathcal{G})}\|v-\alpha\|_{L^2_{-1,-1}(\mathcal{G})} \\ &\leq C\|f\|_{L^2_{1,1}(\mathcal{G})}\|\tilde{v}-\alpha\|_{L^2_{-1,-1}(\mathbf{R}^2)} \\ &\leq C\|f\|_{L^2_{1,1}(\mathcal{G})}|\tilde{v}|_{H^1(\mathbf{R}^2)} \\ &\leq C\|f\|_{L^2_{1,1}(\mathcal{G})}|v|_{H^1(\mathcal{G})}. \end{aligned}$$

$\square$

We are able to address the existence and uniqueness of the solution of the problem (3.1) with $d = 2$ in the energy space $E(\mathcal{G})$.

Theorem 3.10 If $f \in L^2_{1,1}(\mathcal{G})$ and $\sum_{k\in\mathcal{N}} f_k = 0$, then the problem (3.1) with $d = 2$ has a solution $u \in E(\mathcal{G})$, and the

$$|u|_{H^1(\mathcal{G})} \leq C\|f\|_{L^2_{1,1}(\mathcal{G})}. \tag{3.28}$$

The solution is unique in $E(\mathcal{G})$ up to a constant for $s = 1$ and a rigid body motion for $s = d = 2$.

Proof By the property of the bilinear form B

$$|B(u,v)| \leq C\|u\|_{E(\mathcal{G})}\|v\|_{E(\mathcal{G})}$$

and

$$B(u,u) = \|u\|^2_{E(\mathcal{G})},$$

B is continuous. Due to Theorem 3.7, $f \in L^2_{1,1}(\mathcal{G})$ with $\sum_{k\in\mathcal{N}} f_k = 0$ defines a linear functional on $E(\mathcal{G})$, and

$$F(v) = |\langle f, v\rangle_{\mathcal{G}}| \leq C\|f\|_{L^2_{-1,-1}(\mathcal{G})}|v|_{H^1(\mathcal{G})} \leq C\|f\|_{L^2_{-1,-1}(\mathcal{G})}\|v\|_{E(\mathcal{G})}.$$

If a function with zero strain energy is regarded as "zero" in $E(\mathcal{G})$, the energy space $E(\mathcal{G})$ is a Hilbert space. By Lax-Milgram Theorem, there exists a unique solution $u \in E(\mathcal{G})$ such that

$$\|u\|_{E(\mathcal{G})} \leq C\|f\|_{L^2_{1,1}(\mathcal{G})}.$$

Hence (3.23) holds. Owing to the rigidity of the lattices, a function with zero energy is a constant for $s = 1$ and a rigid body motion for $s = d = 2$. Therefore, the solution of the problem (3.8) in two dimensions is unique up to a constant for $s = 1$ or a rigid motion for $s = d = 2$. $\square$

3.3.3 Lattice problems without absolute terms in three dimensions

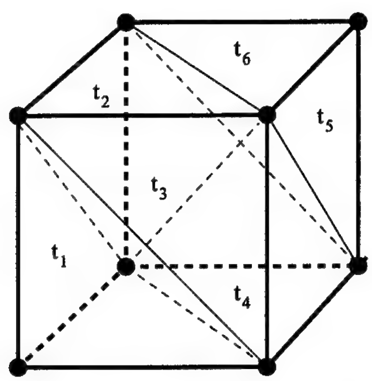
For the existence and uniqueness of solutions on lattice $\mathcal{G}$ in R^3, we have to establish the extension of grid functions on three dimensional lattice $\mathcal{G}$ to whole space R^3.

As in two dimensions, let $\bar{Q}_m$ be the closure of Q_m, and let $\bar{K}_m$ be a set of all nodes in $\bar{Q}_m$. By K_m^V, K_m^E, K_m^F and K_m^I, we denote the subsets of $\bar{K}_m$ for nodes at vertices, on the edges (not including the vertices), on faces (not including nodes on the edges and at vertices), and in the interior of $\bar{Q}_m$, respectively.

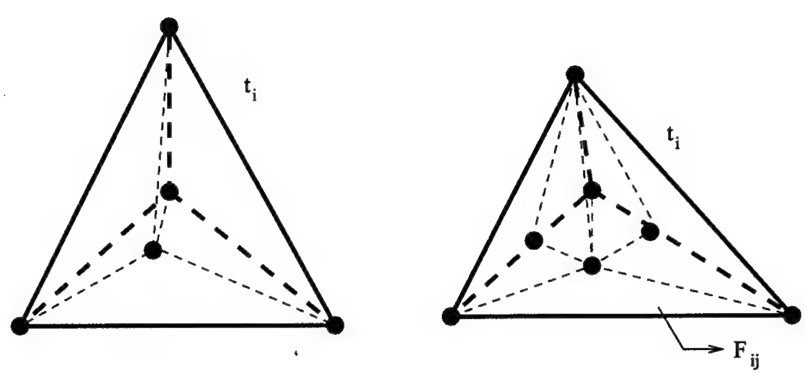
By $\mathcal{T} = \{t_i, 1 \leq i \leq T\}$ we denote a tetrahedral partition of $\bar{Q}_m$ satisfying the conditions :

(T.3) $V_{\mathcal{T}_m} = \bar{K}_m$, where $V_{\mathcal{T}_m}$ denotes a set of all vertices of the partition $\mathcal{T}$;

(T.4) $t_i \cap t_j$ for $i \neq j$ is a vertex, or an edge of t_i, or a face of t_i, or empty.



(a) Initial partition T_m^0 of cell $\bar{Q}_m$ with 6 simplices



(b) Partition of a simplex t_i with a node in the interior

(c) Partition of simplex t_i with nodes on a face F_{ij}

Fig. 3.2 Tetrahedral partition $\mathcal{T}_m$ of cell $\bar{Q}_m$

Each t_i is a simplex with faces $F_{ij}, 1 \leq j \leq 4$. The construction of such a partition can be started with an initial partition $\mathcal{T}_m^0$ of $\bar{Q}_m$ such that (T.4) hold and $V_{\mathcal{T}_m^0} = K_m^V$, shown in Fig. 3.2(a). For the partition $\mathcal{T}_m^0$ there may be some

nodes in the interior of simplices t_is. If a node $x^{(n,\kappa)} \in K_m^I$ is in the interior of t_i, we divide t_i into four smaller simplices by connecting $x^{(n,\kappa)}$ to four vertices of t_i, shown in Fig. 3.2(b). Repeating the process for each node in the interiors of all simplices, we have a partition $\mathcal{T}_m^1$ of $\bar{Q}_m$ for which (T.4) holds and no node $x^{(n,\kappa)} \in \bar{K}_m$ is located in the interiors of all simplices. Note that nodes $x^{(n,\kappa)} \in K_m^I \cup K_m^F \cup K_m^E$ may be located on the closure of faces of t_is in the partition $\mathcal{T}_m^1$. Suppose there are several nodes are on $\bar{F}_{ij}$ which is the closure of F_{ij}. According to the triangular partition of a cell in two dimensions, described in previous subsection, there is a triangular partition $\mathcal{T}_{\bar{F}_{ij}} = \{\tau_l, 1 \le l \le L\}$ of $\bar{F}_{ij}$ such that (T.1) and (T.2) are satisfied. Connecting the vertices of the partition $\mathcal{T}_{\bar{F}_{ij}}$ and the vertex opposite to the face F_{ij}, we divide this simplex t_i into several smaller simplices. If F_{ij} is shared by a pair of simplices, the division can be done in each of them. Carrying this division on each face of simplices t_i and each simplex in the partition $\mathcal{T}_m^1$, we will obtain a desired partition $\mathcal{T}_m$ satisfying (T.3) and (T.4). A combination of the partitions $\mathcal{T}_m$ for all $m \in \mathcal{Z}^3$ form a tetrahedral partition $\mathcal{T}$ of $\mathbf{R}^3$ and $V_{\mathcal{T}} = \cup_{m\in\mathcal{Z}^3}\bar{K}_m$, where $V_{\mathcal{T}}$ is the set of all vertices of $\mathcal{T}$.

As in two dimension, based on such a tetrahedral partition $\mathcal{T}$ of R^3, we can extend a grid function u defined on three dimensional lattice $\mathcal{G}$ to a function $\tilde{u}(x)$ for $x \in R^3$ by a linear interpolation. Let $\phi_i(x)$ be a linear function in t_i which interpolates u at the vertices of t_i, and let $\psi_m(x) = \phi_i(x)$ for $x \in t_i, 1 \le i \le T$, which is a piecewise linear and continuous function in $\bar{Q}_m$, and

$$\begin{aligned} |\psi_m|^2_{H^1(Q_m)} &\le C \sum_{t_i \subseteq \bar{Q}_m} \sum_{x^{(n,\kappa)}, x^{(l,\lambda)} \in t_i} \frac{|u_{n,\kappa} - u_{l,\lambda}|^2}{|x^{(n,\kappa)} - x^{(l,\lambda)}|^2} \\ &\le C \sum_{t_i \subseteq \bar{Q}_m} \sum_{x^{(n,\kappa)}, x^{(l,\lambda)} \in t_i} |u_{n,\kappa} - u_{l,\lambda}|^2. \end{aligned}$$

Let $\tilde{u}(x) = \psi_m(x)$ for $x \in \bar{Q}_m, m \in \mathcal{Z}^3$. Then, $\tilde{u}(x)$ is a continuous and piecewise linear function in R^3, and

$$|\tilde{u}|^2_{H^1(R^3)} \le C \sum_{m\in\mathcal{Z}^3} \sum_{t_i \subseteq \bar{Q}_m} \sum_{x^{(n,\kappa)}, x^{(l,\lambda)} \in t_i} |u_{n,\kappa} - u_{l,\lambda}|^2.$$

Note that vertices of a simplex t_i may not be connected. Arguing as in two dimensions for (3.15), we have

$$|\tilde{u}|^2_{H^1(R^3)} \le C \sum_{m\in\mathcal{Z}^3} \sum_{\kappa\in\mathcal{K}} \sum_{(n,\lambda)\in B_\kappa} |u_{m,\kappa} - u_{m+n,\lambda}|^2 \le C|u|^2_{H^1(\mathcal{G})}.$$

The arguments for the equivalence between norms of u and its extension $\tilde{u}(x)$ in two dimensions can be carried out in the three dimensions. Hence we have the following theorem which is parallel to Theorem 3.8.

Theorem 3.11 Let u be a grid function on a lattice $\mathcal{G}$, and let $\tilde{u}$ be the extension of u by linear interpolation, described as above. Then, $|u|_{H^1(\mathcal{G})} \cong |\tilde{u}|_{H^1(\mathbf{R}^3)}$

and $|u|_{L^2_{\nu,\sigma}(\mathcal{G})} \cong |\tilde{u}|_{L^2_{\nu,\sigma}(R^3)}$, i.e. there are two positive constants independent of u and $\tilde{u}$ such that

$$C_1|u|_{H^1(\mathcal{G})} \le |\tilde{u}|_{H^1(\mathbf{R}^3)} \le C_2|u|_{H^1(\mathcal{G})} \tag{3.29}$$

and

$$C_1|u|_{L^2_{\nu,\sigma}(\mathcal{G})} \le |\tilde{u}|_{L^2_{\nu,\sigma}(\mathbf{R}^3)} \le C_2|u|_{L^2_{\nu,\sigma}(\mathcal{G})}. \tag{3.30}$$

Lemma 3.12 For $v \in E(\mathcal{G})$ there exists a constant α such that

$$\|v-\alpha\|_{L^2_{-1}(\mathcal{G})} \le C|v|_{H^1(\mathcal{G})} \tag{3.26}$$

Proof. Let $\tilde{v}$ be the extension of v described above, and let

$$\alpha = \lim_{r\to\infty} \frac{1}{|S|}\int_S \tilde{v}(r,\theta,\phi)dS.$$

where S is the unit sphere centered at the origin. By Lemma A.10 the above limit exists. Due to Theorem 3.11 and Lemma A.12, we have

$$\|v-\alpha\|_{L^2_{-1}(\mathcal{G})} \le C\int_{\mathbf{R}^3} \frac{|\tilde{v}-\alpha|^2}{r^2}dx \le C\int_{\mathbf{R}^3} |\nabla\tilde{v}|^2 dx \le C|v|_{H^1(\mathcal{G})}.$$

□

Theorem 3.13 If $f \in L^2_1(\mathcal{G})$, and $\sum_{k\in\mathcal{N}} f_k = 0$, then for any $v \in E(\mathcal{G})$,

$$|\langle f,v\rangle_{\mathcal{G}}| = \left|\sum_{k\in\mathcal{N}} f_k v_k\right| \le C\|f\|_{L^2_1(\mathcal{G})}|v|_{H^1(\mathcal{G})} \tag{3.31}$$

Proof. Let $\tilde{v}$ be the extension of v, and let

$$\alpha = \lim_{r\to\infty} \frac{1}{|S|}\int_S \tilde{v}(r,\theta,\phi)dS.$$

as in previous lemma. Then by Cauchy inequality and Lemma 3.5 and Lemma 3.9

$$\begin{aligned} |\langle f,v\rangle_{\mathcal{G}}| &= |\langle f, v-\alpha\rangle_{\mathcal{G}}| \le C\|f\|_{L^2_1(\mathcal{G})}\|v-\alpha\|_{L^2_{-1}(\mathcal{G})} \\ &\le C\|f\|_{L^2_1(\mathcal{G})}\|\tilde{v}-\alpha\|_{L^2_{-1}(R^3)} \le C\|f\|_{L^2_1(\mathcal{G})}|\tilde{v}|_{H^1(R^3)} \\ &\le C\|f\|_{L^2_1(\mathcal{G})}|v|_{H^1(\mathcal{G})}. \end{aligned}$$

□

and due to lemma 2.1, its (energy) norm is equivalent to the semi-norm of $H^1(\mathcal{G})$. Note that $\mathcal{P}_r = span\{e_i, \tau_{i+3}, 1 \le i \le 3\}$ for $s = d = 3$, and $\mathcal{P}_r = \mathcal{P}_c$ denotes the set of constant functions on $\mathcal{G}$ for $s = 1$, which are defined in (2.16).

Theorem 3.14 If $f \in L^2_1(\mathcal{G})$ and $\sum_{k\in\mathcal{N}} f_k = 0$, then the problem (3.8) with $d = 3$ has a solution $u \in E(\mathcal{G})$, and the

$$|u|_{H^1(\mathcal{G})} \le C\|f\|_{L^2_{1,1}(\mathcal{G})}. \tag{3.3}$$

The solution is unique in $E(\mathcal{G})$ up to a constant for $s = 1$ and a rigid body motion for $s = d = 3$.

Proof. It is shown that

$$|B(u,v)| \leq C\|u\|_{E(\mathcal{G})}\|v\|_{E(\mathcal{G})}$$

and

$$B(u,u) = \|u\|^2_{E(\mathcal{G})}.$$

By Theorem 3.13, $f \in L^2_1(\mathcal{G})$ with $\sum_{k\in\mathcal{N}} f_k = 0$ defines a linear functional over $E(\mathcal{G})$, and

$$|F(v)| = |\langle f, v\rangle_{\mathcal{G}}| \leq C\|f\|_{L^2_1(\mathcal{G})}|v|_{H^1(\mathcal{G})} \leq\leq C\|f\|_{L^2_1(\mathcal{G})}\|v\|_{E(\mathcal{G})}.$$

Let functions with zero energy be regarded as a "zero" element in the energy space $E(\mathcal{G})$, then $E(\mathcal{G})$ is a Hilbert space. By Lax-Milgram Theorem, there exists a solution $u \in E(\mathcal{G})$ such that

$$|u|_{H^1(\mathcal{G})} \cong \|v\|_{E(\mathcal{G})} \leq C\|f\|_{L^2_{1,1}(\mathcal{G})}.$$

By the rigidity of the lattices and Proposition 2.1, a function with zero energy is a constant grid function for $s = 1$ and a rigid body motion for $s = d = 3$ on $\mathcal{G}$. Therefore the solution of the problem (3.8) is unique up to a constant for $s = 1$ and a rigid body motion for $s = d = 3$. □

4. UNSTRUCTURED LATTICES IN HALF SPACES

4.1 Setting of unstructured lattices in half spaces $\mathbf{R}^d_+$

For a lattice $\mathcal{G}$ in a upper-half space $\mathbf{R}^d_+ = \{x = (x', x_d) \in \mathbf{R}^d \mid x_d \geq 0\}$, the conditions $(C'.1)$-$(C'.3)$ on connectivity and the assumption on the rigidity are valid. The notations in previous section will be adopted here. In addition we need to precisely characterize unique features of lattices in an half space Let $\Gamma_d = \{x = (x', x_d) \in \mathbf{R}^d \mid x_d = 0\}$ be the hyperplane. For $d = 2$, Γ_2 is the real line and it is consisting of rods and nodes. For $d = 3, \Gamma_2$ is the $x_1 - x_2$ plane and is consisting of polygons for which each edge is a rod and these polygons are faces of the polyhedral cells. By K^b we denote the subset of K in which nodes are on the hyperplane Γ_d, and $K^0 = K \setminus K^b$. K^b is referred as the boundary $\Gamma_{\mathcal{G}}$ of the lattice $\mathcal{G}$ By $\mathcal{N}^b$ we denote the subset of $\mathcal{N}$ such that $x_k \in K^b$ for $k \in \mathcal{N}^b$, and $\mathcal{N}^0 = \mathcal{N} \setminus \mathcal{N}^b$.

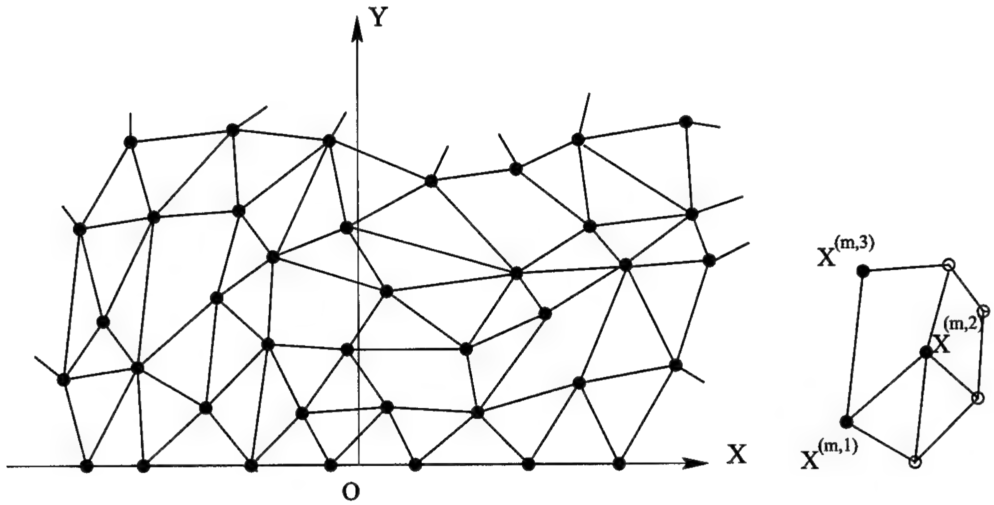
By K^b_m we denote a subset of K_m in which the nodes are on the hyperplane Γ_d, and $K^0_m = K_m \setminus K^b_m$. Further, we denote the subset of $\mathcal{N}_m$ by $\mathcal{N}^b_m$ for the indices of nodes in K^b_m and $\mathcal{N}^0_m = \mathcal{N}_m \setminus \mathcal{N}^b_m$. We may set $m_d = 0$ for those cells $Q_m, m \in \mathcal{Z}^d$ connected to Γ_d. Therefore, K^b_m and $\mathcal{N}^b_m$ are not empty for $m \in \mathcal{Z}^d_+, m_d = 0$, and $K^0_m = K_m$ and $\mathcal{K}^0_m = \mathcal{K}_m$ for all $m \in \mathcal{Z}^d_+, m_d > 0$.

4.2. Boundary Value Problems We consider the boundary value problem

$$-\sum_{\ell\in\mathcal{K}_k} \mathbf{E}^{(k,\ell)} \frac{(u_\ell - u_k)}{|x_\ell - x_k|^2} + \mathbf{C}^k u_k = f_k,\ \forall k \in \mathcal{N}^0 \qquad (4.1a)$$

with boundary condition

$$u_k = 0 \ \text{ for } k \in \mathcal{N}^b \qquad (4.1b)$$



(a) A unstructured lattice in half space R^2_+ (b) An interior cell

Fig. 4.1 An unstructured lattice in half space $\mathbf{R}^2_+$

where

$$\mathbf{E}^{(k,\ell)} = AE^{(k,\ell)} \frac{(x_\ell - x_k)}{|x_\ell - x_k|^2} \frac{(x_\ell - x_k)^\top}{|x_\ell - x_k|^2}$$

and

$$\mathbf{C}^k = diag(C^k_1, C^k_2, \ldots, C^k_d), \quad C^k_l \geq 0, 1 \leq l \leq d.$$

Let $H^1(\mathcal{G})$ and $L^2(\mathcal{G})$ be the discrete Sobolev spaces with the norms

$$\|u\|^2_{L^2(\mathcal{G})} = \sum_{k \in \mathcal{N}^0} |u_k|^2 \tag{4.2}$$

and

$$\|u\|^2_{H^1(\mathcal{G})} = |u|^2_{H^1(\mathcal{G})} + \|u\|^2_{L^2(\mathcal{G})} \tag{4.3a}$$

where $|u|_{H^1(\mathcal{G})}$ is the semi-norm,

$$|u|^2_{H^1(\mathcal{G})} = \sum_{k \in \mathcal{N}^0} \sum_{\ell \in \mathcal{K}_k} \frac{|u_\ell - u_k|^2}{|x_\ell - x_k|^2}. \tag{4.3b}$$

By $H^1_0(\mathcal{G})$, we denote the subspace of $H^1(\mathcal{G})$, in which functions vanish at the boundary $\Gamma_{\mathcal{G}} = K^b$.

The corresponding variational equation

$$B(u, v) = F(v), \ \forall v \in H^1_0(\mathcal{G}) \tag{4.4}$$

are associated with

$$B(u, v) = \sum_{k \in \mathcal{N}^0} \left\{ \sum_{\ell \in \mathcal{K}_k} \langle \frac{1}{2} \mathbf{E}^{(k,\ell)} \frac{u_\ell - u_k}{|x_\ell - x_k|}, \frac{v_\ell - v_k}{|x_\ell - x_k|} \rangle + \langle \mathbf{C}^k u_k, v_k \rangle \right\} \tag{4.5a}$$

is a bilinear form on $H^1_0(\mathcal{G}) \times H^1_0(\mathcal{G})$, and

$$F(v) = \sum_{k \in \mathcal{N}^0} \langle f_k, v_k \rangle. \tag{4.5b}$$

is a linear functional on $H_0^1(\mathcal{G})$.

The energy of the lattice $\mathcal{G}$ is defined as

$$G(u) = B(u,u) = \sum_{k\in\mathcal{N}^0}\{\sum_{\ell\in\mathcal{K}_k}\langle\frac{1}{2}\mathbf{E}^{(k,\ell)}\frac{u_\ell - u_k}{|x_\ell - x_k|}, \frac{u_\ell - u_k}{|x_\ell - x_k|}\rangle + \langle\mathbf{C}^k u_k, u_k\rangle\} \quad (4.6)$$

As analogue of the Theorem 3.3 for unstructured lattice problem, we have the existence and uniqueness for the boundary value problems (4.1).

Theorem 4.1 If $\mathbf{C}^k \not\equiv 0$ in the sense that there exists a node $x_{k_m} = x^{(m,\kappa_m)}$ in each cell such that $\mathbf{C}^{k_m} \neq 0$, the problem (3.1) has a unique solution $u \in H_0^1(\mathcal{G})$.

4.3 Problems Without Absolute Terms

There is no essential difference to address the existence and uniqueness of solution for problem (3.6) in entire spaces and the boundary value problem (4.1) in half spaces if the problems associated with absolute terms. We should focus the treatment for problem without absolute terms in half space, which is essentially different from those for the problem in entire space. Without the absolute term the boundary value problem is to find a $u \in E_0(\mathcal{G}) = \{v \in E(\mathcal{G}) \mid v = 0 on \Gamma_{\mathcal{G}}\}$ such that

$$B(u,v) = F(v), \ \forall v \in E_0(\mathcal{G}) \quad (4.7)$$

where b is a bilinear form on $E_0(\mathcal{G}) \times E_0(\mathcal{G})$,

$$B(u,v) = \sum_{k\in\mathcal{N}^0}\sum_{\ell\in\mathcal{K}_k}\langle\frac{1}{2}\mathbf{E}^{(k,\ell)}\frac{u_\ell - u_k}{|x_\ell - x_k|}, \frac{v_\ell - v_k}{|x_\ell - x_k|}\rangle \quad (4.8a)$$

and F is a linear functional on $E_0(\mathcal{G})$,

$$F(v) = \langle f, v\rangle_{\mathcal{G}} = \sum_{k\in\mathcal{N}}\langle f_k, v_k\rangle. \quad (4.9b)$$

For the problem (4.7), the energy space $E(\mathcal{G})$ does not coincide the $H^1(\mathcal{G})$, and $E(\mathcal{G})$ is not embedded in $L^2(\mathcal{G})$. The "energy norm" $\|u\|_{E(\mathcal{G})}$ is equivalent to the semi-norm of $H^1(\mathcal{G})$. Due to Lemma 2.1 and the boundary condition (4.1b) we have the uniqueness of the solution in $E_0(\mathcal{G})$.

Theorem 4.2 If $u \in E_0(\mathcal{G})$ is a solution of the problem (4.1) over the lattice $\mathcal{G}$ which satisfies the rigidity assumption, then it is unique.

Proof If there is another solution $v \in E_0(\mathcal{G})$, then $\|u - v\|_{E(\mathbf{R}_+^d)} = 0$. Since the lattice $\mathcal{G}$ is rigid under the assumption, by Lemma 2.1, $u - v$ is a rigid body motion. Since $(u - v)\mid_{\Gamma_{\mathcal{G}}} = 0$, hence $u = v$. □

The theorem tells the space $E_0(\mathcal{G})$ is a normed space and the energy norm is a norm for the space $E_0(\mathcal{G})$. For the existence of the solution, we need to find a weighted space in which function f is linear functional over $E_0(\mathcal{G})$. To

this end, we introduce a weighted space $L^2_\nu(\mathcal{G})$ with the norm

$$\|u\|^2_{L^2_\nu(\mathcal{G})} = \sum_{k\in\mathcal{N}^0} \left(1+|x_{k,d}|^2\right)^\nu |u_k|^2$$

with a real number ν, where $x_{k,d}$ is the d-th coordinate of the node x_k which is the distance form the node x_k to the hyperplane Γ_d. We shall write $L^2_0(\mathcal{G}) = L^2(\mathcal{G})$. Further, we need to modify the $H^1(\mathcal{G})$ by introducing the space $\tilde{H}^1(\mathcal{G})$ with the norm

$$\|u\|_{\tilde{H}^1(\mathcal{G})} = \left\{\|u\|^2_{L^2_{-1}(\mathcal{G})} + |u|^2_{H^1(\mathcal{G})}\right\}^{1/2}.$$

and the space

$$\tilde{H}^1_0(\mathcal{G}) = \{u \in \tilde{H}^1(\mathcal{G}) \mid u_{m,\kappa} = 0 \text{ on } \Gamma_{\mathcal{G}}\}$$

To prove $E_0(\mathcal{G}) \subset L^2_{-1}(\mathcal{G})$, which is essential for the existence of solution, we have to extend the grid functions on $\mathcal{G}$ to continuous functions in $\mathbf{R}^d_+$ by the linear interpolation in one, two and three dimensions. For $d = 1, u \in E_0(\mathcal{G})$ is extended to $\tilde{u}$ such that

$$\tilde{u}(x_k) = u_k \text{ for } k \in \mathcal{N}. \tag{4.10}$$

For $d = 2, 3$ the extension of u based on the triangular partition has been described in previous section. can be carried out here, and (4.10) holds at every node of $\mathcal{G}$ for all dimension. Arguing as in proof of Theorem 3.8, we have the equivalence of norms of u and $\tilde{u}(x)$.

Theorem 4.3 Let u be a grid function on the lattice $\mathcal{G}$, and let $\tilde{u}$ be the extension of u by a linear interpolation, described as above. Then $|u|_{H^1(\mathcal{G})} \cong |\tilde{u}|_{H^1(\mathbf{R}^2)}$, and $\|u\|_{L^2_{\nu,\sigma}(\mathcal{G})} \cong \|\tilde{u}\|_{L^2_{\nu,\sigma}(\mathbf{R}^2)}$, i.e. there are two positive constants C_1 and C_2 independent of u and $\tilde{u}$ such that

$$C_1|u|_{H^1(\mathcal{G})} \leq |\tilde{u}|_{H^1(\mathbf{R}^2)} < C_2|u|_{H^1(\mathcal{G})} \tag{4.11}$$

and

$$C_1\|u\|_{L^2_{\nu,\sigma}(\mathcal{G})} \leq \|\tilde{u}\|_{L^2_{\nu,\sigma}(\mathbf{R}^2)} < C_2\|u\|_{L^2_{\nu,\sigma}(\mathcal{G})} \tag{4.12}$$

The equivalence between norms of u and its extension $\tilde{u}$leads to a desired embedding lemma.

Lemma 4.4 $E_0(\mathcal{G}) \subset L^2_{-1}(\mathcal{G})$,and for $u \in E_0(\mathcal{G})$, there holds

$$\|u\|_{L^2_{-1}(\mathcal{G})} \leq C|u|_{H^1(\mathcal{G})} \tag{4.12}$$

with C independent of u.

Proof Let $\tilde{u}(x)$ be the extension of u by a linear interpolation described above. Due to Theorem 4.3, $\tilde{u}(x) \in E_0(R^d_+)$. By Lemma B.1,

$$\|\tilde{u}\|_{L^2_{-1}(R^2_+)} \leq C|\tilde{u}|^2_{H^1(R^2_+)}.$$

which together with Theorem 4.3 imply (4.12) immediately. □

The following lemma indicates that the energy norm $\|u\|_{E(\mathcal{G})}$ is a norm of the energy space $E_0(\mathcal{G})$.

Lemma 4.5 The space $E_0(\mathcal{G})$ is equivalent to the space $\tilde{H}_0^1(\mathcal{G})$, and for $u \in E_0(\mathcal{G})$ there holds

$$\|u\|_{E(\mathcal{G})} \cong \|u\|_{\tilde{H}^1(\mathcal{G})} \cong |u|_{H^1(\mathcal{G})}. \tag{4.13}$$

□

Lemma 4.6 If $f \in L_1^2(\mathcal{G})$, then for any $v \in E_0(\mathcal{G})$ there holds

$$|\langle f, v\rangle| \leq C\|f\|_{L_1^2(\mathcal{G})}\|v\|_{E(\mathcal{G})}. \tag{4.14}$$

Proof By Schwarz inequality, there holds

$$|\langle f, v\rangle| \leq C\|f\|_{L_1^2(\mathcal{G})}\|v\|_{L_{-1}^2(\mathcal{G})}.$$

Due to Lemma 4.4-4.5 we have

$$|\langle f, v\rangle| \leq C\|f\|_{L_1^2(\mathcal{G})}|v|_{H^1(\mathcal{G})} \leq C\|f\|_{L_1^2(\mathcal{G})}\|v\|_{E(\mathcal{G})}.$$

□

The above lemmas lead to the existence and uniqueness of solution for the problem (4.7).

Theorem 4.7 For $f \in L_1^2(\mathcal{G})$, the problem (4.7) for $d = 1, 2, 3$ has a unique solution $u \in E_0(\mathcal{G})$, and

$$\|u\|_{E(\mathcal{G})} \leq C\|f\|_{L_1^2(\mathcal{G})}. \tag{4.15}$$

Proof Due to Lemma 4.5, we have for $u, v \in E_0(\mathcal{G})$

$$\begin{aligned} |B(u,v)| &\leq C|u|_{H^1(\mathcal{G})}|v|_{H^1(\mathcal{G})} \\ &\leq C\|u\|_{\tilde{H}^1(\mathcal{G})}\|v\|_{\tilde{H}^1(\mathcal{G})} \end{aligned}$$

and

$$B(u,u) = |u|_{H^1(\mathcal{G})}^2 \geq D\|u\|_{\tilde{H}^1(\mathcal{G})}^2.$$

By Lemma 4.6, f defines a linear functional $F(v)$ on $E_0(\mathcal{G})$, and for $v \in E_0(\mathcal{G})$

$$|F(v)| = |\langle f, v\rangle| \leq C\|f\|_{L_{-1}^2(\mathcal{G})}\|v\|_{E(\mathcal{G})} \leq C\|f\|_{L_{-1}^2(\mathcal{G})}\|v\|_{\tilde{H}^1(\mathcal{G})}.$$

By Lax-Milgram Theorem, there exists a unique solution $u \in \tilde{H}_0^1(\mathcal{G})$ such that

$$\|u\|_{\tilde{H}^1(\mathcal{G})} \leq C\|f\|_{L_{-1}^2(\mathcal{G})}$$

which with Lemma 4.6 leads to the assertion of the theorem. □

Remark 4.1 The boundary Γ_ϕ can be a very general curve, and the function $x_d = \phi(x')$ is required to be a piecewise continuous function and allow to approach ∞ as $|x'| \to \infty$, e.g. $x_2 = x_1^2$ for $d = 2$. The weight $(1 + x_d^2)^\nu$ may be modified to $(1 + d(x)^2)^\nu$ where $d(x)$ is the distance from x to the boundary Γ_ϕ, for the detalis of such a modification we refer to [2].

CONCLUSION

The lemmas and theorems in previous sections indicates that a mathematical framework for problems of unbounded lattices has been established. In this framework we are able to prove the existence and uniqueness of solution for the

problems of unstructured lattice in entire and half spaces. This is a significant progress in research of lattice problems.

We have derived the appropriate function spaces which the data f (e.g. external force) belongs to, and proved the existence of solutions for the lattice problems without absolute terms in entire and half spaces if the date f is given in these function spaces. The function spaces and approaches to prove the existence theorems are quite different in one,two and three dimension. Meanwhile, because of the Dirichlet boundary condition for lattice problems without absolute terms in half spaces, the function spaces and approaches to prove the existence theorems are the same or very similar in one, two and three dimension. This reflects the fundamental differences between problems without boundary and the boundary value problems. The results reported here and in forth coming papers could be further extended and generalized as indicated in various remarks and comments in previous sections. Although the theorems are proved for truss problems, it can be utilized or generalized to problems of general lattices such as plates shells, and three dimensional solid.

Based on the progresses we have made in the past years, our research on lattices should be further carried on. The focus of the research at next stage will be the analysis of complicated models and design of effective computation. A direct extension of our research results is the investigation on the existence and uniqueness of solutions for general lattice problems without absolute terms in unbounded domains, associate with various structures such as plate, shell and 3-dimensional solid. This is an important issue and has been an not- well-unanswered problems for last 2 decades. The extension of grid functions by a linear interpolation for rigid and non-triangular or non-tetrahedral lattices implies an effective numerical approach for unstructured lattices, i.e. multigrid method. To this end we shall define partial differential equations(PDE) associate with boundary conditions or no boundary conditions, which is equivalent to corresponding lattice problems. This equivalent PDE is not homogenized one because homogenization is not applicable to the unstructured lattice, and can be solved on fine and coarse grids. The discretization of the PDE on the grid of lattices is the original problem of lattice. The approach has been investigated for unstructured lattices in bounded domains [4], it could be carried for unbounded lattices.

If the scale of cells of lattices is very small, we have to deal with a multi-scale problem which is one of the most popular topics of modeling and computing in modern applied and computational mathematics and engineering. The framework we have built for the problem without absolute terms will help us to analyze various aspects of such multi-scale problems.

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APPENDIX A

In this appendix we will address the existence and uniqueness of solution of the Dirichlet problem of Laplace equation in $\mathbf{R}^d, d = 2, 3$. We are interested in this problem because it is similar to the problem (3.1) of unstructured lattices in entire spaces without absolute term. In particular, some embedding results play a key role for proving the existence and uniqueness of solution for such lattice problems. We seek a solution $u \in H(\mathbf{R}^d)$ of the variational problem :

$$B(u,v) = F(v)\ \forall v \in H(\mathbf{R}^d) \qquad (A.1a)$$

with

$$B(u,v) = \int_{\mathbf{R}^d} \nabla u \cdot \nabla v dx,\ \ u, v \in H(\mathbf{R}^d) \qquad (A.1b)$$

and

$$F(v) = \int_{\mathbf{R}^d} f v dx,\ \ v \in H(\mathbf{R}^d) \qquad (A.1c)$$

where $d = 2, 3$, and

$$H(\mathbf{R}^d) = \left\{ u \mid u \in H^1_{local}(\mathbf{R}^d), |u|_{H^1(\mathbf{R}^d)} < \infty \right\}$$

Here $|u|_{H^1(\mathbf{R}^d)}$ is the semi-norm of the Sobolev space $H^1(\mathbf{R}^d)$ involving only the first derivatives.

A.1 In two dimensions

Let $\tilde{H}^1(\mathbf{R}^2)$ be a closure of C^∞ functions with the norm

$$\|u\|^2_{\tilde{H}^1(\mathbf{R}^2)} = \int_S |u|^2 dx + \int_{S^c} \frac{|u|^2}{r^2 \log^2 r} dx + \int_{\mathbf{R}^2} |\nabla u|^2 dx \qquad (A.2)$$

where (r, θ) are the polar coordinates, the disc $S = \{x \in \mathbf{R}^2 | r = |x| \leq 2\}$ and $S^c = \mathbf{R}^2 \setminus S$. By $L^2_{\nu,\sigma}(\mathbf{R}^2)$ we denote a weighted space with the norm

$$\|u\|^2_{L^2_{\nu,o}(\mathbf{R}^2)} = \int_S |u|^2 dx + \int_{S^c} |u|^2 r^{2\nu} \log^{2\sigma} r dx. \qquad (A.3)$$

Then we have

$$\|u\|^2_{\tilde{H}^1(\mathbf{R}^2)} = \|u\|^2_{L^2_{-1,-1}(\mathbf{R}^2)} + |u|^2_{H^1(\mathbf{R}^2)}.$$

We further introduce

$$H_0(\mathbf{R}^2) = \left\{ u \in H(\mathbf{R}^2) \mid \int_\Gamma u ds = 0, \right\}$$

and

$$H_\Gamma(S^c) = \left\{ u \mid \int_{S^c} |\nabla u|^2 dx < \alpha, u\,|_\Gamma = 0 \right\}$$

where $\Gamma = \partial S$, and we define a quotient space

$$\tilde{\tilde{H}}^1(\mathbf{R}^2) = \tilde{H}^1(\mathbf{R}^2)/P_0 \qquad (A.4a)$$

with the norm

$$\|u\|_{\tilde{\tilde{H}}^1(\mathbf{R}^2)} = \inf_{\alpha \in P_0} \|u - \alpha\|_{H^1(\mathbf{R}^2)} \qquad (A.4b)$$

where P_0 is the set of all real numbers. Since $P_0 \subset \tilde{H}^1(\mathbf{R}^2)$, the quotient space is well defined. For the existence and uniqueness of the solution of the problem (A.1) with $d = 2$, we need to establish the equivalence between $\|u\|_{\tilde{\tilde{H}}^1(\mathbf{R}^2)}$ and $|u|_{H^1(\mathbf{R}^2)}$ and to show that $F(v)$ in $(A.1c)$ defines a linear functional over $\tilde{H}^1(\mathbf{R}^2)$.

Theorem A.1 If $u \in H_0(\mathbf{R}^2)$, then

$$\int_S |u|^2 dx + \int_{S^c} \frac{|u|^2}{r^2 \log^2 r} dx \le C \int_{\mathbf{R}^2} |\nabla u|^2 dx \tag{A.5}$$

with constant C independent of u.
Theorem A.1 is a major theorem in this section, we need several lemmas to prove it.

Lemma A.2 Let $u(t)$ be a function on $(0, \infty)$ satisfying $u(2) = 0$ and

$$\int_2^\infty \left| \frac{du}{dt} \right|^2 t dt < \infty.$$

Then

$$\int_2^\infty \frac{|u|^2}{t \log^2 t} dt \le C \int_2^\infty \left| \frac{du}{dt} \right|^2 v dt \tag{A.6}$$

Proof. Due to Theorem 1.14 of [29]

$$\int_2^\infty |u|^2 w dt \le C_L \int_2^\infty \left| \frac{du}{dt} \right|^2 v dt$$

where $w = \frac{1}{t \log^2 t}$, $v = t$ and $C_L = \sup_{2 \le t \le \infty} F_L(t)$, with

$$F_L(t) = \left(\int_t^\infty w dt \right)^{\frac{1}{2}} \left(\int_2^t v^{-1} dt \right)^{\frac{1}{2}}.$$

Note that

$$\int_t^\infty w dt = \int_2^\infty \frac{dt}{t \log^2 t} = \frac{1}{\log t}$$

and

$$\int_2^t v^{-1} dt = \int_2^t \frac{1}{t} dt = \log t - \log 2.$$

Then

$$F_L(t) = \sqrt{\frac{\log t - \log 2}{\log t}} \le 1 \text{ for } 2 \le t \le \infty,$$

and (A.6) follows immediately.

Lemma A.3 If $u \in H_\Gamma(S^c)$, then

$$\int_{S^c} \frac{|u|^2}{r^2 \log^2 r} dx \le C \int_{S^c} |\nabla u|^2 dx \tag{A.7}$$

Proof. Note that $u(r,\theta)\mid_{r=2}=0$, which implies by Lemma A.2 that

$$\int_2^\infty |u|^2 \frac{1}{r\log^2 r} dr \le C \int_2^\infty \left|\frac{\partial u}{\partial r}\right|^2 r dr$$

with constant C independent of θ. Integrating with respect to θ from 0 to 2π we have (A.6).

Lemma A.4 Let $u \in H^1(S)$, where $H^1(S)$ is a usual Sobolev space on a bounded domain S. Then the following norm

$$|||u|||_{H^1(S)} = \left\{ |u|^2_{H^1(S)} + \left|\int_\Gamma uds\right|^2 \right\}^{\frac{1}{2}} \tag{A.8}$$

is equivalent to the norm, $\|u\|_{H^1(S)}$.

Proof. Obviously $|||u|||_{H^1(S)}$ is a norm to $H^1(S)$. Note that

$$\left|\int_\Gamma uds\right|^2 \le C\int_\Gamma |u|^2 ds \le C\|u\|_{H^{\frac{1}{2}}(\Gamma)} \le C\|u\|_{H^1(S)}$$

which implies

$$|||u|||_{H^1(S)} \le C_1 \|u\|_{H^1(S)}.$$

We need to show that

$$\|u\|_{H^1(S)} \le C_2 |||u|||_{H^1(S)}.$$

If it is false, there exists a sequence $u_j \in H^1(S), j = 1,2,\ldots$ such that $\|u_j\|_{H^1(S)} = 1$, and

$$|||u_j|||^2_{H^1(S)} = |u_j|^2_{H^2(S)} + \left|\int_\Gamma uds\right|^2 \to 0 \text{ as } j \to \infty$$

Since $H^1(S) \subset\subset L^2(S)$, there exists a subsequence denoted by u_j again, which is a Cauchy sequence in $L^2(S)$. Since $|u_j|_{H^2(S)} \to 0$ as $j\to\infty$, $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $H^1(S)$ as well. Hence $\lim_{j\to\infty} u_j = u_0$ in $H^1(S)$. This implies that $D^\alpha u_0 = \lim_{j\to\infty} D^\alpha u_j = 0$ for $|\alpha| = 1$. Therefore u_0 is a constant in S. Note that

$$\left|\int_\Gamma (u_j - u_0)\, ds\right|^2 \le C\int_\Gamma (u_j-u_0)^2\, ds \le C\|u_j - u_0\|^2_{H^1(S)} \to 0 \text{ as } j\to\infty,$$

which leads to

$$\int_\Gamma u_0 ds = \lim_{j\to\infty} \int u_j ds = 0.$$

Hence $u_0 \equiv 0$ in S. It contradicts the fact that $\|u_0\|_{H^1(S)} = \lim_{j\to\infty} \|u_j\|_{H^1(S)} = 1$. Thus the lemma is proved.

Lemma A.5 If $u \in H_0(\mathbf{R}^2)$, u has the Fourier series on Γ:

$$u(2,\theta) = \sum_{k=1}^\infty a_k \cos k\theta + b_k \sin k\theta$$

and

$$\sum_{k=1}^{\infty}\left(a_k^2+b_k^2\right)k \le C|u|^2_{H^1(S)} \le C|u|^2_{H^1(\mathbf{R}^2)}$$

Proof. Since $\int_\Gamma u ds = 0$, the coefficient $a_0 = 0$, and

$$U(2,\theta) = \sum_{k=1}^{\infty}\left(a_k \cos k\theta + b_k \sin k\theta\right).$$

Then by the trace theorem we have

$$\sum_{k=1}^{\infty}\left(a_k^2+b_k^2\right)k \approx \|u\|^2_{H^{1/2}(\Gamma)} \le C\|u\|^2_{H^1(S)}$$

by Lemma A.4

$$\le C|||u|||^2_{H^1(S)} = C|u|^2_{H^1(S)} \le C|u|^2_{H^1(\mathbf{R}^2)}.$$

We now prove Theorem A.1.

Proof of Theorem A.1. For $u \in H_0\left(\mathbf{R}^2\right)$ we can find a harmonic function u_1 such that $(u-u_1)|_\Gamma = 0$ and $\int_{S^c}|\nabla u_1|^2 dx < \infty$. Let $u_2 = u - u_1$. Then $u_2 \in H_0\left(S^c\right)$, and by Lemma A.3

$$\int_{S^c}|u_2|^2\frac{1}{r^2\log^2 r}dx \le C\int_{S^c}|\nabla u_2|^2 dx \tag{A.9}$$

Since $u_1(2,\theta) = u(2,\theta), u_1$ has a Fourier series on Γ

$$u_1(2,\theta) = \sum_{k=1}^{\infty}\left(a_k \cos k\theta + b_k \sin k\theta\right)$$

with $a_0 = 0$. Because u_1 is harmonic in S^c,

$$u_1(r,\theta) = \sum_{k=1}^{\infty}\left(a_k \cos k\theta + b_k \sin k\theta\right)\left(\frac{2}{r}\right)^k$$

and

$$\begin{aligned}\int_{S^c}|u_1|^2\frac{1}{r^2\log^2 r}rdrd\theta &\le C\sum_{k=1}^{\infty}\left(a_k^2+b_k^2\right)4^k\int_2^{\infty}\frac{rdr}{r^{2k+2}\log^2 r}\\ &\le C\sum_{k=1}^{\infty}\left(a_k^2+b_k^2\right)k\end{aligned}$$

by Lemma A.5

$$\le C|u|^2_{H^1(S)} \tag{A.10}$$

Note that $u_2|_\Gamma = 0$ and $\Delta u_1 = 0$ in S^c, which implies that

$$\int_{S^c}\nabla u_1 \nabla u_2 dx = \int_\Gamma u_2\frac{\partial u_1}{\partial n}ds - \int_{S^c}u_2\Delta u_1 dx = 0.$$

Hence

$$|u|^2_{H^1(S^c)} = |u_1|^2_{H^1(S^c)} + |u_2|^2_{H^1(S^c)},$$

which together with (A.9) - (A.10) leads to

$$\begin{aligned} |u|^2_{L^2_{-1,-1}(S^c)} &\le C\left(\|u_1\|^2_{L^2_{-1,1}(S^c)} + \|u_2\|^2_{L^2_{-1,1}(S^c)}\right) \\ &\le C\left(|u|^2_{H^1(S)} + |u_2|^2_{H^1(S^c)}\right) \\ &\le C|u|^2_{H^1(\mathbf{R}^2)}. \end{aligned} \tag{A.11}$$

We next shall show that

$$\|u\|_{L^2(S)} \le C|u|_{H^1(S)} \tag{A.12}$$

Let v_1 be harmonic in S and $v_1|_\Gamma = u|_\Gamma$, and let $v_2 = u - v_1$. Since $v_2|_\Gamma = 0$, by Lemma A.4

$$\|v_2\|_{L^2(S)} \le \|v_2\|_{H^1(S)} \le C\,|v_2|_{H^1(S)}\,. \tag{A.13}$$

Since v_1 is harmonic and $\int_\Gamma v_1 ds = \int_\Gamma u ds = 0$

$$v_1(r,\theta) = \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta)\left(\frac{r}{2}\right)^k$$

and

$$|v_1|^2_{L^2(S)} \le C\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right) k \le C|u|^2_{H^1(S)}. \tag{A.14}$$

Since v_1 is harmonic and v_2 vanishes on Γ, we have, by the argument above for u_1 and u_2, that

$$|v|^2_{H^1(S)} = |v_1|^2_{H^1(S)} + |v_2|^2_{H^1(S)}$$

which together with (A.13) – (A.14) leads to (A.12) immediately. A combination of (A.11) and (A.12) yields (A.5).

Corollary A.6 The norm $\|u\|_{\hat{H}^1(\mathbf{R}^2)}$ is equivalent to $|u|_{H^1(\mathbf{R}^2)}$, and the spaces $\hat{H}^1(\mathbf{R}^2)$ and $H(\mathbf{R}^2)$ are equivalent.

Remark A.1. The weight function $w(x) = 1$ in S, and $w(x) = r^{-2}\log^{-2} r$ in S^c. S^c excludes the origin and unit circle. We may select others weight, e.g., $w(x) = (1+r^2)^{-1}\log^{-2}(2+r)$ for all $x \in \mathbf{R}^2$. It is essential for the selection of the weight that

$$|w| = O\left(|x|^{-2}\log^{-2}|x|\right) \text{ for large } |x|.$$

Also S can be selected to any bounded domain with Lipschitz boundary, and it is not necessary to be the disk centered at the origin and with radius 2.

Theorem A.7 If $f \in L^2_{1,1}(\mathbf{R}^2)$, and $\int_{\mathbf{R}^2} f dx = 0$, then for any $v \in H(\mathbf{R}^2)$

$$\left|\int_{\mathbf{R}^2} f v dx\right| \le C\|f\|_{L^2_{1,1}(\mathbf{R}^2)}|v|_{H^1(\mathbf{R}^2)} \tag{A.15}$$

with constant C independent of u.

Proof. Let $v \in H(\mathbf{R}^2)$, and let $\alpha = \int_\Gamma v ds$. Then $(v - \alpha) \in H^0(\mathbf{R}^2)$, and

$$\begin{aligned} \left|\int_{\mathbf{R}^2} f v dx\right| &= \left|\int_{\mathbf{R}^2} f(v-\alpha) dx\right| \\ &\leq \|f\|_{L^2_{1,1}(\mathbf{R}^2)} \|v-\alpha\|_{L^2_{-1,-1}(\mathbf{R}^2)} \end{aligned}$$

by Theorem A.1

$$\leq C\|f\|_{L^2_{1,1}(\mathbf{R}^2)} |v|_{H^1(\mathbf{R}^2)}.$$

Theorem A.8 If $f \in L^2_{1,1}(\mathbf{R}^2)$ and $\int_{\mathbf{R}^2} f dx = 0$, then the problem (A.1) with $d = 2$ has a solution $u \in H(\mathbf{R}^2)$, and

$$|u|_{H^1(\mathbf{R}^2)} \leq C\|f\|_{L^2_{1,1}(\mathbf{R}^2)}. \tag{A.16}$$

The solution is unique up to a constant.

Proof. Due to Corollary A.6 we have

$$|B(u,v)| \leq |u|_{H^1(\mathbf{R}^2)} |v|_{H^1(\mathbf{R}^2)} \leq C\|u\|_{\tilde{\tilde{H}}^1(\mathbf{R}^2)} \|v\|_{\tilde{H}^1(R)}$$

and

$$|B(u,u)| = |u|^2_{H^1(\mathbf{R}^2)} = \|u\|^2_{\tilde{\tilde{H}}^1(\mathbf{R}^2)}.$$

By Lemma A.2 there holds

$$\begin{aligned} |F(v)| &\leq C\|f\|_{L^2_{1,1}(\mathbf{R}^2)} |v|_{H^1(\mathbf{R}^2)} \\ &= C\|f\|_{L^2_{1,1}(\mathbf{R}^2)} \|v\|_{\tilde{\tilde{H}}^1(\mathbf{R}^2)} \end{aligned}$$

By Lax-Milgram lemma there exist a unique solution $u \in \tilde{\tilde{H}}^1(\mathbf{R}^2)$ such that

$$\|u\|_{\tilde{\tilde{H}}^1(\mathbf{R}^2)} \leq C\|f\|_{L^2_{1,1}(\mathbf{R}^2)}$$

which together with Corollary A.6 leads to the assertion of the theorem.

Corollary A.9. If $f \in L^2_{\nu,0}(\mathbf{R}^2)$ with $\nu > 1$ the problem (A.15) has a unique solution in $H(\mathbf{R}^2)$ up to a constant, and

$$\|u\|_{H(\mathbf{R}^2)} \leq C\|f\|_{L^2_{\nu,0}(\mathbf{R}^2)}.$$

A.2 In three dimensions

In three dimension we shall deal with the existence and uniqueness of the solution of variational problem (A.1) in the way different from the approach in two dimensions, e.g. we will not introduce the quotient spaces.

Let

$$\tilde{u}(r) = \frac{1}{|S|}\int_S u(r,\theta,\phi) dS = \frac{1}{|S|}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi) sin\theta \, d\theta \, d\phi \tag{A.17}$$

where S denotes the unit sphere, and (r, θ, ϕ) are the spheric coordinates.

Lemma A.10 If $u \in H(\mathbf{R}^3)$, then $\lim_{r\to\infty} \tilde{u}(r) = A$ exists.

Proof. For $r \geq 1$,

$$\tilde{u}(r) = \frac{1}{|S|} \int_S \int_1^r \frac{\partial u(t,\theta,\phi)}{\partial t} dt dS + v(1)$$

Let $r_j, j = 1, 2, \ldots$ be an arbitrary sequence with $\lim_{j\to\infty} r_j = \infty$. For $r_j > r_i$ we have

$$\begin{aligned} |\tilde{u}(r_j) - \tilde{u}(r_i)| &= \frac{1}{|S|} \int_S \int_{r_i}^{r_j} \frac{\partial u(t,\theta,\phi)}{\partial t} dt dS \\ &\leq C \left(\int_S \int_{r_i}^{r_j} |\frac{\partial u(t,\theta,\phi)}{\partial t}|^2 t^2 dt dS \right)^{1/2} \left(\int_{r_i}^{r_j} t^{-2} dt \right)^{1/2} \\ &\leq C \left(\frac{1}{r_i} - \frac{1}{r_j} \right)^{1/2} |u|_{H^1(\mathbf{R}^3)} \end{aligned}$$

This implies that $\{\tilde{u}(r_j)\}_{j=1}^{\infty}$ is a Cauchy sequence and that $\tilde{u}(r_j)$ converges to the same limit A for all sequence $\{r_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} r_j = \infty$. Therefore, $\lim_{r\to\infty} \tilde{u}(r)$ exists, and $\lim_{r\to\infty} \tilde{u}(r) = A$.

Lemma A.11 Let $\tilde{u}(r)$ be given in (A.17) and $A = \lim_{r\to\infty} \tilde{u}(r)$. Then

$$\int_0^{\infty} |\tilde{u}(r) - A|^2 dr \leq C \int_0^{\infty} |\tilde{u}'(r)|^2 r^2 dr \leq C |u|_{H^1(\mathbf{R}^3)} \qquad (A.18)$$

Proof Let $w(r) = \tilde{u}(r) - A$. Then $\lim_{r\to\infty} w(r) = 0$. By Hardy inequality 330 [19], we have

$$\int_0^{\infty} |w(r)|^2 dr \leq C \int_0^{\infty} |w'(r)|^2 r^2 dr$$

which is the first inequality of (A.18), The second one follows from

$$\int_0^{\infty} |\tilde{u}'(r)|^2 r^2 dr \leq \int_0^{\infty} \int_S |\frac{\partial u(r,\theta,\phi)}{\partial r}| r^2 dr dS.$$

□

Theorem A.12 If $|u|_{H^1(\mathbf{R}^3)} < \infty$, then there exist a constant α such that

$$\int_{\mathbf{R}^3} \frac{|u-\alpha|^2}{r^2} dx \leq C \int_{\mathbf{R}^3} |\nabla u|^2 dx.$$

Proof. Let $\alpha = A = \lim_{r\to\infty} \tilde{u}(r)$. Then

$$\int_{\mathbf{R}^3} \frac{|u-\alpha|^2}{r^2} dx \leq C \left(\int_{\mathbf{R}^3} \frac{|\tilde{u}(r) - \alpha|^2}{r^2} |dx + \int_{\mathbf{R}^3} \frac{|u - \tilde{u}(r)|^2}{r^2} |dx \right) \qquad (A.19)$$

For the first term on the right hand side of (A.19), we have by Lemma A.11

$$\int_{\mathbf{R}^3} \frac{|\tilde{u}(r) - \alpha|^2}{r^2} dx \leq \int_S \int_0^{\infty} |\tilde{u}(r) - \alpha|^2 dr dS \leq C |u|^2_{H^1(\mathbf{R}^3)}. \qquad (A.20)$$

For the second term we write

$$u(r,\theta,\phi) - \tilde{u}(r) = \frac{1}{|S|} \int_S (u(r,\theta,\phi) - u(r,\theta',\phi'))\, sin\theta'\, d\theta'\, d\phi'$$

and

$$u(r,\theta,\phi) - u(r,\theta',\phi') = u(r,\theta,\phi) - u(r,\theta',\phi) + u(r,\theta',\phi) - u(r,\theta',\phi').$$

Note that

$$\begin{aligned}|u(r,\theta,\phi) - u(r,\theta',\phi)|^2 &= |\int_{\theta'}^{\theta} \frac{\partial u(r,\tau,\phi)}{\partial \tau} d\tau|^2 \\ &\leq C \int_0^{\pi} |\frac{\partial u(r,\theta,\phi)}{\partial \theta}|^2 d\theta\end{aligned}$$

which implies

$$\int_S |u(r,\theta,\phi) - u(r,\theta',\phi)|^2 dS \leq \int_S |\frac{\partial u(r,\theta,\phi)}{\partial \theta}|^2 dS$$

and

$$\int_{\mathbf{R}^3} \frac{|u(r,\theta,\phi) - u(r,\theta',\phi)|^2}{r^2} dx \leq C \int_{\mathbf{R}^3} |\nabla u|^2 dx. \qquad (A.21)$$

Similarly, it can be shown that

$$\int_{\mathbf{R}^3} \frac{|u(r,\theta',\phi) - u(r,\theta',\phi')|^2}{r^2} dx \leq C \int_{\mathbf{R}^3} |\nabla u|^2 dx. \qquad (A.22)$$

A combination of (A.19)-(A.22) leads to (A.18). □

Theorem A.13 If $f \in L_1^2(\mathbf{R}^3)$, and $\int_{\mathbf{R}^3} f dx = 0$, then for any $v \in H^(\mathbf{R}^3)$,

$$|\int_{\mathbf{R}^3} fv dx| \leq C \|f\|_{L_1^2(\mathbf{R}^3)} |v|^2_{H^1(\mathbf{R}^3)}. \qquad (A.23)$$

Here and thereafter D_R denote a ball centered at the origin with radius R, and

$$\|f\|^2_{L_1^2(\mathbf{R}^3)} = \int_{\mathbf{R}^3} (1+r^2)|f|^2 dx.$$

Proof. Since $\int_{\mathbf{R}^3} f dx = 0$

$$\int_{\mathbf{R}^3} fv dx = \int_{\mathbf{R}^3} f(v - A) dx$$

with $A = \lim_{r\to\infty} \frac{1}{|S|} \int_S v(r,\theta,\phi) dS$. By Lemma A.11,

$$\begin{aligned}|\int_{\mathbf{R}^3} fv dx| &\leq C \left\{\int_{\mathbf{R}^3} |f|^2(1+r^2) dx\right\}^{1/2} \left\{\int_{\mathbf{R}^3} \frac{|v|^2}{(1+r^2)} dx\right\}^{1/2} \\ &\leq C \|f\|_{L_1^2(\mathbf{R}^3)} |v|_{H^1(\mathbf{R}^3)}.\end{aligned}$$

□

The next theorem addresses the existence and uniqueness of the solution of the variational equation (A.1) with $d = 3$ in the case that $\int_{\mathbf{R}^3} f dx = 0$.

Theorem A.14 If $f \in L_1^2(\mathbf{R}^3)$, and $\int_{\mathbf{R}^3} f dx = 0$, then the variational problem (A.1) with $d = 3$ has a solution $u \in H(\mathbf{R}^3)$,

$$|u|_{H^1(\mathbf{R}^3)} \leq C \|f\|_{L_1^2(\mathbf{R}^3)}, \qquad (A.24)$$

and the solution is unique up to a constant.

Proof. It is easy to see that for $u, v \in H(\mathbf{R}^3)$

$$\begin{aligned} |B(u,v)| &\leq C|u|_{H^1(\mathbf{R}^3)}|v|_{H^1(\mathbf{R}^3)}, \\ |B(u,u)| &= |u|^2_{H^1(\mathbf{R}^3)} \end{aligned}$$

and by Theorem A.11, f is a linear functional on $H(\mathbf{R}^3)$, and

$$|\int_{\mathbf{R}^3} fvdx| \leq C\|f\|_{L^2_1(\mathbf{R}^3)}|v|^2_{H^1(\mathbf{R}^3)}.$$

If functions v in $H(\mathbf{R}^3)$ with $|v|^2_{H^1(\mathbf{R}^3)} = 0$ is regarded as a "zero" element, then the space $H^1(\mathbf{R}^3)$ is a Hilbert space with $\|u\|_{H(\mathbf{R}^3)} = |u|_{H^1(\mathbf{R}^3)}$. By Lax-Milgram Theorem, there exist a unique solution $u \in H(\mathbf{R}^3)$, and

$$\|u\|_{H(\mathbf{R}^3)} \leq C\|f\|_{L^2_1(\mathbf{R}^3)}.$$

which leads to the assertion of the theorem. $\square$

In three dimensions we noticed that there are functions $u_0 \in H(\mathbf{R}^3)$ satisfies the equation

$$-\Delta u_0 = f_0.$$

with $\int_{\mathbf{R}^3} f_0 dx \neq 0$, and $u_0(x) = O(\frac{1}{|x|})$ as $|x| \to \infty$. For example,

$$u_0 = \begin{cases} \dfrac{3r^2}{16\pi} & \text{in } D_1 \\ \dfrac{3}{8\pi}(\dfrac{3}{2} - \dfrac{1}{r}) & \text{in } D_1^c \end{cases}$$

and

$$f = -\Delta u_0 = \begin{cases} \dfrac{3}{4\pi} & \text{in } D_1 \\ 0 & \text{in } D_1^c \end{cases}$$

where D_1 is a unit ball and $D_1^c = \mathbf{R}^3 \setminus D_1$. Note that $\int_{\mathbf{R}^3} f_0 dx = 1$ and f_0 has compact support D_1, and $|u_0|_{H^1(\mathbf{R}^3)} < \infty$. Hence in the case that $\int_{\mathbf{R}^3} fdx \neq 0$ we seek a weak solution $u = c_0 u_0 + w \in H(\mathbf{R}^3)$ such that w satisfies the variational equation :

$$B(w,v) = \int_{\mathbf{R}^3} (f - c_0 f_o)vdx \;\forall v \in H(\mathbf{R}^3) \tag{A.25}$$

with $c_0 = \int_{\mathbf{R}^3} fdx$. Since $z = (f - c_0 f_0) \in L^2_1(\mathbf{R}^3)$ and $\int_{\mathbf{R}^3} zdx = 0$, by Theorem A.13, w exists, and

$$\|w\|_{H(\mathbf{R}^3)} \leq C\|z\|_{L^2_1(\mathbf{R}^3)} \leq C(|c_0| + \|f\|_{L^2_1(\mathbf{R}^3)}).$$

Therefore we have a corollary on the existence and uniqueness of the solution for variational equation (A.1) in three dimensions.

Corollary A.15 If $f \in L_1^2(\mathbf{R}^3)$, and $\int_{\mathbf{R}^3} f dx = c_0 \neq 0$, then there exists a solution $u \in H(\mathbf{R}^3)$ such that $u = c_0 u_0 + w$, where $w \in H(\mathbf{R}^3)$ is the solution of the variational equation (A.25),

$$|u|_{H^1(\mathbf{R}^3)} \leq C(\|f\|_{L_1^2(\mathbf{R}^3)} + |\int_{\mathbf{R}^3} f dx|).$$

and the solution is unique up to a constant.

APPENDIX B

In this appendix we will address the existence and uniqueness of solution of the Dirichlet problem of Laplace equation in half space or in domains of half-space type. We are interested in this problem because it is similar to the problem (3.1) of unbounded lattices with boundary condition and without absolute term. In particular, an embedding result contained in Lemma A.1 play a key role for proving the existence and uniqueness of solution for such lattice problems.

Let $R_+^d = \{x = (x', x_d) \in R^d \mid x_d \geq 0\}$ be the upper half space, and let $\Gamma_d = \{x = (x', x_d) \in R^d \mid x_d = 0\}$ be its boundary. By $H^1(R_+^d)$ we denote the Sobolev space over R_+^d, and we introduce the energy space :

$$E(R_+^d) = \left\{u \mid u \in H_{local}^1(R_+^d), |u|_{H^1(R_+^d)} < \infty\right\}$$

where $|u|_{H^1(R_+^d)}$ is the semi-norm of the Sobolev space $H^1(R_+^d)$ involving only the first derivatives. $E(R_+^d)$ and $H^1(R_+^d)$ are not equivalent, and the energy norm $|u|_{E(R_+^d)}$ is equal to the semi-norm $|u|_{H^1(R_+^d)}$. By $E_0(R_+^d)$ we denote the subspace of $E(R_+^d)$ in which functions vanish on Γ_d.

We now address the existence and uniqueness of the solution of the variational problem : seek $u \in E_0(R_+^d), d = 2, 3$ such that

$$B(u, v) = F(v), \ \ \forall v \in E_0(R_+^d) \tag{B.1a}$$

with

$$B(u, v) = \int_{R_+^d} \nabla u \cdot \nabla v dx, \ \ u, v \in E_0(R_+^d) \tag{B.1b}$$

and

$$F(v) = \int_{R_+^d} f v dx, \ \ v \in E_0(R^d) \tag{B.1c}$$

where $E_0(R_+^d) = \{u \in E(R_+^d) \mid u = 0 \text{ on } \Gamma_d\}$, $\Gamma_d = \{x = (x', x_d) \in R^d \mid x_d = 0\}$ is the hyperplane, and

$$E(R_+^d) = \left\{u \mid u \in H_{local}^1(R_+^d), |u|_{H^1(R_+^d)} < \infty\right\}$$

Here $|u|_{H^1(R_+^d)}$ is the semi-norm of the Sobolev space $H^1(R_+^d)$ involving only the first derivatives, and $R_+^d = \{x = (x', x_d) \in R^d \mid x_d \geq 0\}$ is the upper-half space. $E(R_+^d)$ is called the energy space with energy norm $|u|_{E(R_+^d)} = B(u, v)^{1/2}$ which is equivalent to the semi-norm $|u|_{H^1(R_+^d)}$. $E_0(R_+^d)$ is its sub space of functions vanishing at the boundary Γ_d, which is not equivalent to the Sobolev $H^1(R_+^d)$, and is not embedded in $L^2(R_+^d)$. Therefore, the problem (3.1) may have no solution in $E_0(R_+^d)$, e.g. if $f \in L^2(R_+^d)$. Hence we have to find a space for f which has stronger topology than $L^2(R_+^d)$ for existence of solutions for the problem (B.1). To this end, we introduce a weighted space $L_\nu^2(R_+^d)$ furnished with a norm

$$||u||_{L_\nu^2(R_+^d)}^2 = \int_{R_+^d} |u|^2 (1 + x_d^2)^\nu dx \tag{B.2}$$

We further modify the $H^1(R^d_+)$ by introducing the space $\tilde{H}^1(R^d_+)$ with its norm

$$\|u\|^2_{\tilde{H}^1(R^d_+)} = \|u\|^2_{L^2_{-1}(R^d_+)} + |u|^2_{H^1(R^d_+)}. \tag{B.3}$$

and the space $\tilde{H}^1_0(R^d_+) = \{\tilde{H}^1(R^d_+) \mid u = 0 \text{ on } \Gamma_d\}$.

We are interested in such a problem because it is parallel to the problem (4.1) of unbounded lattices without absolute term, and a embedding result contained in Lemma B.1 play a key role for proving the existence and uniqueness of solution for such lattice problems.

Lemma B.1 $E_0(R^d_+) \subset L^2_{-1}(R^d_+)$, and for $u \in E_0(R^d_+)$

$$\|u\|^2_{L^2_{-1}1(R^d_+)} \leq C\|u\|_{E(R^d_+)} \tag{B.4}$$

with constant C independent of u.

Proof For $u \in E_0(R^d_+)$, there holds

$$u(x) = u(x', x_d) = \int_0^{x_d} \frac{\partial u(x',t)}{\partial t} dt$$

Selecting a $\sigma \in (0,1)$, we have by Schwarz inequality

$$\begin{aligned} |u(x)|^2 &= |\int_0^{x_d} \frac{\partial u(x',t)}{\partial t} dt|^2 \\ &\leq C \int_0^{x_d} t^{-\sigma} dt \int_0^{x_d} |\frac{\partial u(x',t)}{\partial t}|^2 t^\sigma dt \\ &= C x_d^{1-\sigma} \int_0^{x_d} |\frac{\partial u(x',t)}{\partial t}|^2 t^\sigma dt \end{aligned} \tag{B.5}$$

which leads to

$$\begin{aligned} \|u\|_{L^2_{-1}(R^d_+)} &\leq C \int_{R^d_+} \left(\frac{x_d^{1-\sigma}}{1+x_d^2} \int_0^{x_d} |\frac{\partial u(x',t)}{\partial t}|^2 t^\sigma dt\right) dx \\ &= C \int_{R^{d-1}} \{\int_0^\infty \left(\frac{x_d^{1-\sigma}}{1+x_d^2} \int_0^{x_d} |\frac{\partial u(x',t)}{\partial t}|^2 t^\sigma dt\right) dx_d\} dx'. \end{aligned} \tag{B.6}$$

Note that

$$\begin{aligned} & \int_0^\infty \frac{x_d^{1-\sigma}}{1+x_d^2} (\int_0^{x_d} |\frac{\partial u(x',t)}{\partial t}|^2 t^\sigma dt) dx_d \\ = \;& \int_0^\infty |\frac{\partial u(x',t)}{\partial t}|^2 t^\sigma (\int_t^\infty \frac{x_d^{1-\sigma}}{1+x_d^2} dx_d) dt \\ \leq \;& C \int_0^\infty |\frac{\partial u(x',t)}{\partial t}|^2 t^\sigma \left(\int_t^\infty x_d^{-1-\sigma} dx_d\right) dt \\ = \;& C \int_0^\infty |\frac{\partial u(x',t)}{\partial t}|^2 dt. \end{aligned} \tag{B.7}$$

Combining (B.5)-(B.7) we have

$$\|u\|_{L^2_{-1}(R^d_+)} \leq C|u|_{H^1(R^d_+)} = C\|u\|_{E(R^d_+)}.$$

□

As a consequence of Lemma B.1, we have the following corollary which indicates that the space $E_0(R^d_+)$ is a normed space and the energy norm is a norm of the space $E_0(R^d_+)$, which is equivalent to the norm of $\tilde{H}^1(R^d_+)$.

Corollary B.2 The space $E_0(\mathcal{G})$ is equivalent to the space $\tilde{H}_0^1(R_+^d)$, and $\|u\|_{E(\mathcal{G})}$ is a norm of the space $E_0(R_+^d)$. For $u \in E_0(R_+^d)$, there holds

$$\|u\|_{E(R_+^d)} \cong |u|_{H^1(R_+^d)} \cong \|u\|_{\tilde{H}^1(R_+^d)}. \qquad (B.8)$$

Lemma B.3 If $f \in L_1^2(R_+^d)$, then for any $v \in E_0(R_+^d)$, there holds

$$|\langle f, v\rangle| \leq C\|f\|_{L_1^2(R_+^d)}\|v\|^2_{E(R_+^d)}. \qquad (B.9)$$

Proof By Schwarz inequality, there holds

$$|\langle f, v\rangle| \leq \|f\|_{L_1^2(R_+^d)}\|v\|^2_{L^2_{-1}(R_+^d)}$$

which together with (B.4) and (B.8) leads to (B.9) immediately. □

We now are able to address the existence and uniqueness of solution for the problem (B.1) in the framework of the space $\tilde{H}_0^1(R_+^d)$.

Theorem B.4 For $f \in L_1^2(R_+^d)$, the problem (B.1) has a unique solution $u \in E_0(R_+^d)$, and

$$\|u\|^2_{E(R_+^d)} \leq C\|f\|_{L_1^2(R_+^d)}. \qquad (B.10)$$

Proof By Corollary B.2, there hold for $u, v \in E_0(R_+^d)$

$$|B(u,v)| \leq C\|u\|_{E(R_+^d)}\|v\|_{E(R_+^d)} \leq C\|u\|_{\tilde{H}^1(R_+^d)}\|v\|_{\tilde{H}^1(R_+^d)}$$

and

$$B(u,u) = \|u\|^2_{E(R_+^d)} \geq D\|u\|_{\tilde{H}^1(R_+^d)}.$$

Due to Lemma B.3, $f \in L_1^2(R_+^d)$ defines a linear functional on $E_0(R_+^d)$, and

$$|F(v)| \leq C\|f\|_{L_1^2(R_+^d)}\|v\|_{E(R_+^d)} \leq C\|f\|_{L_1^2(R_+^d)}\|v\|_{\tilde{H}^1(R_+^d)}.$$

By Lax-Milgram Theorem, the variational problem (B.1) has a unique solution $u \in \tilde{H}_0^1(R_+^d)$, and

$$\|u\|_{\tilde{H}^1(R_+^d)} \leq C\|f\|_{L_1^2(\mathcal{G})}.$$

Corollary B.2 implies the existence and uniqueness of the solution in $E_0(R_+^d)$ for the problem (B.1), and (B.10) follows from (B.8).

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